On the evolution and breakup of slender drops in an extensional flow

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The evolution of the shape of an elongated drop embedded in an extensional flow is studied in the framework of slender body theory. The external flow has a weak but not neglected inertia. The problem is governed by three dimensionless parameters: the capillary number, the external Reynolds number, and the viscosity ratio between the drop and the external fluid, and exhibits a multiplicity of stationary shapes with only one being stable. Evolution of the drop surface from initial shapes was studied when the flow intensity was either kept constant or subjected to a sudden change. It was shown that the dynamics of the shape evolution can lead to a breakup of the drop or to a stable stationary shape. Two modes of breakup are revealed: an indefinite elongation and a center pinching mode. The former appears when the viscous forces dominate the inertia effects, with the typical case being that of a creeping flow. The latter breakup mode takes over in the presence of inertia when the drop viscosity diminishes with the extreme example being that of an inviscid drop. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.3701373]

I. INTRODUCTION

The problem of deformation and breakup of a single drop suspended in another fluid undergoing shear or extensional flow has been the subject of many theoretical and experimental studies. These studies, which reveal that drop deformation can span a wide domain from a slightly perturbed sphere up to a slender body with pointed ends, were summarized in two reviews by Rallison1 and Stone.2 In this report we shall focus however on slender bodies only which, in creeping flow, are obtained when the external viscous force is much greater than the surface tension force and when the viscosity of the drop is much smaller than the viscosity of the external fluid. In terms of the dimensionless governing numbers, we require high capillary numbers, $Ca \gg 1$, and small viscosities ratios, $\lambda \ll 1$. Thus, such slender drops are generally observed when (relative) low viscosity drops are suspended in high viscosity liquids such as polymer melts, foods, biological materials and glasses.

A theoretical model for a slender drop in an axisymmetric extensional creeping flow ($Re = 0$) was presented by Taylor,3 Buckmaster,4,5 Acrivos and Lo6 and others. Accordingly, the stationary slender drop has a parabolic shape with pointed ends. The critical capillary number needed for breakup increases as the viscosity ratio decreases, and for the case of a bubble or an inviscid drop ($\lambda = 0$), a stable shape is always possible, and breakup is not predicted. Hinch and Acrivos7 studied a two-dimensional extensional flow, a case where the drop cross-section is not circular. The evolution of a slender inviscid drop was studied by Hinch,8 while Sherwood9 treated the case of a non-linear extensional flow; their analyses shall be closely followed in this report. Howell and Siegel10 studied the case of a slender non-axisymmetric drop, while the effect of an insoluble surfactant on the
surface of an inviscid slender drop was theoretically investigated by Booty and Siegel. Favelukis and Nir and Favelukis et al. included power-law non-Newtonian effects outside and inside the drop, respectively.

Acrivos and Lo extended the creeping flow theory by including a small amount of inertia to the external fluid (Re ≪ 1). The analysis predicts that, as inertia increases, drop breakup is facilitated. Furthermore, contrary to the creeping flow case, it was found that an inviscid drop (λ = 0) can be broken as well. Note, though, that Brady and Acrivos, who studied the effects of combined external and internal inertia, concluded that internal inertia can be neglected when predicting drop breakup. An interesting result from the theory is the existence of multiple (stable and unstable) stationary solutions. Recently, Favelukis et al. reported a stability analysis to small disturbances that confirmed and expanded Acrivos and Lo inertia effect findings. This linear stability analysis indicates that the drop shape may depart from the steady, yet unstable, state but it cannot predict the evolution process as time progresses. Furthermore, when the drop assumes an evolution process leading to its disintegration, breakup may be caused via more than one mechanism, e.g., center pinching, indefinite elongation or tip streaming.

Several questions arise. Since many stationary states exist, including a stable one, the drop shape may assume a dynamic evolution from an unstable state to the stable one. On the other hand it may drift into a trajectory directing it to breakup. The physical conditions that direct the shape into either dynamics, e.g., the viscosity ratio, flow strength, inertia effect intensity and initial configuration, are of interest. In either case the trajectory of the temporal evolution is of interest as well. A comprehensive experimental study of the deformation and breakup of drops in a two-dimensional extensional flow was reported by Bentley and Leal. The deformation and breakup results for low viscosity drops cannot be directly compared with the predictions made for axisymmetric flow since the respective typical capillary numbers are significantly different. Yet, as will be shown herewith, for cases of diminishing inertia effects there is a qualitative agreement between the breakup mechanism observed by Bentley and Leal and the one predicted in this work. Multiplicity of breakup mechanisms was also observed by Sherwood who studied the case of a drop in a non-linear extensional creeping flow. The selection of such mechanisms will depend on the physical parameters mentioned above as well.

The above questions suggest that a study of the dynamic evolution and breakup mechanisms of a slender drop is important and, hence, it is the subject of this report. In Sec. II, we present the formulation of the problem, the physical groups and scaling, and the dynamic equations to be solved. In Sec. III, we describe the method of solution and address the evolution dynamics from stationary states and the break-up mechanisms involved. Section IV presents a study of dynamics due to changes in the physical parameters, in particular the flow intensity, and explores the conditions for evolution to steady shapes or to breakup of the drop. This is followed by concluding remarks in Sec. V.

II. FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS

Consider a slender drop, with a local radius $R(z,t)$ and a half-length $L(t)$, located at the origin of a cylindrical coordinate system (see Fig. 1).

FIG. 1. A slender drop in a simple extensional flow: $R(z,t)$ is the local radius and $L(t)$ is the half-length of the drop.
The undisturbed motion in the continuous phase, far away from the drop, is the axisymmetric extensional flow,

\[ v_r = -\frac{1}{2} Gr, \quad v_z = Gz, \quad G > 0. \]  

The compressibility of the two fluids is neglected. Neglecting inertia forces inside (but not outside) the drop and rendering the pressure, all the lengths and the time dimensionless with respect to the characteristic stress outside the drop (\( \mu G \)), the radius of a sphere of an equal volume (\( a \)) and \( 1/G \), respectively, Acrivos and Lo\(^6\) obtained the governing equations for the shape of a slender Newtonian drop in a Newtonian fluid, \( R(z,t) \), and for the internal dimensionless pressure at the center of the drop, \( P(0,t) \). These equations contain three dimensionless parameters: the Reynolds number for the external flow and the capillary number, \( \lambda = \frac{\mu_{in}}{\mu} \), \( Re = \frac{\rho Ga^2}{\mu} \), \( Ca = \frac{\mu Ga}{\sigma} \), where \( \mu_{in} \) is the viscosity of the drop, \( \rho \) and \( \mu \) are the density and viscosity of the external fluid, respectively, and \( \sigma \) is the surface tension. Favelukis et al.\(^5\) suggested the use of following rescaled variables: \( y = R Ca \) and \( x = z/Ca^2 \), having the order of magnitude of 1, that reduce the governing equations to the form

\[
\frac{\partial y}{\partial t} + x \frac{\partial y}{\partial x} - y \left( \frac{P(0,t)}{2} - 1 + \frac{1}{4} f_2 x^2 \right) + \frac{1}{2} = 4 f_2^6 \int_0^1 \frac{1}{y} \left( \frac{x}{y} + \frac{1}{3} \frac{\partial}{\partial t} \int_0^x y^2 \, dx \right) \, dx,
\]

\[
\int_0^{x_k} y^2 \, dx = \frac{2}{3},
\]

containing only two dimensional parameters: The inertia strength of the flow: \( f_1 = Ca Re^{1/4} \) and the viscous strength of the flow: \( f_2 = Ca \lambda \). Equations (3) and (4) are subject to an initial drop shape, \( y(x,0) \), and the requirement that the radius at the end of the drop vanishes, \( y(x_L, t) = 0 \) with \( x_L = L/Ca^2 \). At steady state, the radius at the center of the drop will be expressed as \( y(0) = 1/(2\nu) \), where \( \nu = -1 + P(0)/2 \) and \( P(0) \) is the steady pressure at \( x = 0 \), see Eq. (3). Recall that, in order to obtain a slender drop (\( R/L \ll 1 \)) the following conditions must be met: \( Ca^3 \gg 1, \lambda \ll 1 \), \( Ca \ll 1 \), and \( Re Ca \ll 1 \). In terms of the small aspect ratio of the drop, \( \varepsilon = R/L \ll 1 \), the theory holds for small viscosity ratios, \( \lambda \sim \varepsilon^2 \), large capillary numbers, \( Ca \sim \varepsilon^{-1/3} \), and small Reynolds numbers, \( Re \sim \varepsilon^{-1/3} \).

Following Hinch,\(^8\) the pressure at the center of the drop can be obtained by combining Eqs. (3) and (4) and the vanishing time derivative of the drop volume, to give

\[
P(0,t) = 1 + \frac{\int_0^{x_k} y dx - \frac{1}{2} f_1^4 \int_0^{x_k} x^2 y^2 dx - \frac{4}{3} f_2^6 \int_0^{x_k} \left[ \int_0^x \frac{1}{y} \left( \frac{x}{y} + \frac{1}{3} \frac{\partial}{\partial t} \int_0^x y^2 \, dx \right) \right] \, dx}{\int_0^{x_k} y^2 \, dx}.
\]

The unknown pressure at the center of the drop can be further eliminated by differentiating Eq. (3) with respect to \( x \), yielding

\[
2 \frac{\partial^2 y}{\partial x \partial t} - \frac{2}{y} \frac{\partial y}{\partial x} \frac{\partial y}{\partial t} + 2x \frac{\partial^2 y}{\partial x^2} - \frac{2}{y} \left( \frac{\partial y}{\partial x} \right)^2 + \left( 2 - \frac{1}{y} \right) \frac{\partial y}{\partial x} - f_1^4 xy
\]

\[
= 8 f_2^6 \left( \frac{x}{y} + \frac{1}{3} \frac{\partial}{\partial t} \int_0^x y^2 \, dx \right).
\]
Finally, following Favelukis et al., we define a dimensionless parameter $h$, characterizing the relative strength of inertia,

$$
h = \frac{f_1^4}{f_2^6} = \frac{Re}{\lambda Ca^2}.
$$

Note that the parameter $h$ spans a wide domain. When $h \to 0$ the creeping flow limit is obtained, while the case $h \to \infty$ denotes an inviscid drop under the effect of external inertia. Note also that the latter case does not correspond to high inertia since, as it was demonstrated above, the slender body analysis required an asymptotically small Reynolds number, $Re \sim \varepsilon^{4/3}$.

## III. EVOLUTION DYNAMICS FROM STATIONARY SHAPES

In this section, we describe our method to solve the unsteady problems and explore the dynamics emerging from stationary states. The structure and stability of the various stationary states is shortly reviewed and the dynamics of shape evolution emerging from these states is followed. In all the runs and figures the initial shapes of all evolution patterns are calculated using the previous theory (Favelukis et al.).

### A. An inviscid drop ($\lambda = 0$)

For the case of an inviscid drop with inertia ($h \to \infty, f_1 \neq 0, f_2 = 0$), a case where the pressure in the drop is uniform, it is suitable to employ Eqs. (3)–(5). We suggest a solution in terms of an expansion about the center of the inviscid drop,

$$
y(x, t, f_1) = \sum_{k=0}^{\infty} y_k(t, f_1)x^{2k}.
$$

At stationary states, explicit expressions for the coefficients in Eq. (8) have been obtained by Acrivos and Lo. The solution also admits a closed form (Favelukis et al.). The deformation curve of $x_L$ vs. $f_1$ consists of multiple lobes and that each lobe contains two branches separated by a bifurcation turning point and argued that, for the first lobe, along the lower branch, the deformation is stable, while the upper branch is unstable. The other lobes were found to be unstable. The first lobe of the deformation curve $f_1$ versus $\nu$ is shown in Fig. 5.

A substitution of Eq. (8) into Eq. (3) results in the following system for the function coefficients $y_k(t_1)$:

$$
\frac{dy_0}{dt} = \frac{1}{2} \left[-1 + (P - 2) y_0\right]; \quad k = 0,
$$

$$
\frac{dy_k}{dt} = \frac{1}{4} \left[f_1^4 y_{k-1} + (2P - 8k - 4) y_k\right]; \quad k \geq 1.
$$

For the numerical simulations, the infinite series in Eq. (8) was truncated at some $k = K$ and substituted into Eq. (5) resulting, in the case of an inviscid drop, $f_2 = 0$, in an explicit expression of the pressure $P$ as a function of the coefficients $y_0$ to $y_K$. The set of equations given by Eqs (9) and (10) was solved numerically making use of Mathematica. Note that, for the case of a creeping flow ($f_1 = 0$), Hinch reported that the numerical solution becomes unstable if the integral in the denominator of (5) is replaced by the exact value 2/3. Such behavior was also observed in this study. We found that with such replacement the volume is not conserved. Thus, Eq. (5) was used as is without changing the denominator. The accuracy of the solution depends on the number of terms $K$ that are used in the power series of Eq. (8), and is checked by the conservation of volume in Eq. (4). We found that with $K = 7$ (8 terms), the solution is already correct to three significant digits.

We next follow the evolution of the shape starting at a stationary one. If the initial stationary shape is located along the stable branch ($2 < \nu < 2.54$), no dynamic change in the drop is evident. On the other hand, Fig. 4(a) shows the evolution of the shape of the inviscid drop when it is originally
positioned at the unstable branch ($2.54 < \nu < 4$) of the first lobe, see also Table I. The chosen parameters are $\nu = 3.0, f_1 = 0.196, x_L = 39.6$. The shape is unstable to infinitesimal perturbations and, as is depicted in Fig. 4(a), a numerical perturbation of the initial shape eventually grows and the form of the drop in the upper unstable branch changes with time until breakup occurs in a mechanism of center pinching. There is a natural incubation time for the dynamics to be visible which, obviously, depends on the accuracy of the calculation of the initial unstable stationary shape. The more accurate is the initial shape the longer time is required for the evolution to commence and for the inviscid drop to break. Hence, no particular values of time appear in Fig. 4(a).

B. A viscous drop ($\lambda \neq 0$)

For the case of a viscous drop with or without inertia ($f_2 \neq 0$), it proved more convenient to employ the differentiated version of the dynamic equation, Eq. (6). Here, as above, we suggest an expansion of the form

$$y(x,t,f_1,f_2) = \sum_{k=0}^{\infty} y_k(t,f_1,f_2) x^{2k}. \quad (11)$$

Substituting this expansion into Eq. (6), we get

$$\frac{dy_0}{dt} = - y_0 \left[ hf_2^6 y_0^2 + 2 \left( 4 f_2^6 + y_1 \right) - 4 y_0 \left( 2 y_1 + \frac{dy_1}{dt} \right) \right]; \quad k = 0, \quad (12)$$

where $h$ is defined in Eq. (7). For cases where $k \geq 1$, the respective expressions are cumbersome and are not presented here explicitly. However, similar to the above equation, the equation for $y_k$ involves terms of higher index, i.e., $y_{k+1}$ and $dy_{k+1}/dt$.

To compute the coefficients $y_k(t,f_1,f_2)$ the infinite set is truncated at some term $k = K$, and we solve the first $K + 1$ equations assuming $y_{K+1} = 0$. The obtained results were compared with those obtained for increased $K$, to ensure that a convergence with the desired accuracy is achieved. $K = 10$ was found to be sufficient to preserve the volume at an accuracy of $10^{-3} \cdot 10^{-2}$. The solution of the resulting ODE system was carried out numerically making use of Mathematica 7. For initial conditions we used stationary shapes (see Favelukis et al.\textsuperscript{15}).

We next study the evolution dynamics of stationary shapes at various locations on the first lobe for various cases of relative inertia intensity. Table I summarizes the bifurcation points that separate the stable and unstable branches of the stationary shapes (Favelukis et al.\textsuperscript{15}). It refers to the first lobe, $2 < \nu < 4$ and $h > 0$, for flows with inertia and for the single lobe, $2 < \nu < \infty$ and $h = 0$, in the creeping flow case. As the contribution of external inertia increases, $h$ or $f_1$ increases, $f_{2,max}$ decreases, and drop breakup is enhanced.

In the first series of numerical experiments we examined drops located on the two branches of the first lobe and simulated the shape evolution (see Favelukis et al.\textsuperscript{15} for the structure of this lobe for various values of $h$). It was evident that when the initial position is located at a point on the lower branch (for any value of $h$), no dynamic change in the drop shape occurs. However, when the initial shape is located on the upper branch of the first lobe the drop breaks up. In Fig. 2 we show...
the evolution of a viscous drop, originally located at an arbitrary point, \( \nu = 3.0 \), on the unstable branch of the first lobe \( (\nu_{cr} < \nu = 3.0 < 4) \), at different values of \( h \). Here \( \nu_{cr} \), the value of \( \nu \) at the bifurcation turning point (the ‘breakup’ point), is a function of \( h \) and is given in Table I. The results show that the shape of the drop in the upper unstable branch changes with time until the drop breaks in a mechanism that depends on the relative strength of the external inertia. For low values of inertia (low values of \( h \), including the case of creeping flow with \( h = 0 \)), the breakup mechanism is by indefinite elongation. On the other hand, at relatively larger values of inertia (high values of \( h \)), the breakup mechanism is by center pinching.

Similar to the case of the inviscid drop where \( h \rightarrow \infty \) (see Fig. 4(a)), since the incubation time during the onset of evolution depends on the accuracy of the calculations, no specific values are indicated in the figure. Note also, that as the value of \( h \) is increased, the shape of the drop evolves from a “parabolic” long shape to a more “rounded” short shape before breakup. Our numerical simulations indicate that, when the evolution commences at \( \nu = 3.0 \), the transition from a breakup by indefinite elongation to the center pinching mode occurs at about \( h \approx 38 (f_2 = 0.134) \). Note that the same types of breakup mechanisms were obtained by Sherwood\(^9\) for the case of a slender drop in a non-linear extensional creeping flow. There, the non-linear term in the undisturbed flow plays a similar role to that of the non-linear inertia effect.

IV. MODES OF EVOLUTION DUE TO CHANGES OF PHYSICAL PARAMETERS

In this section, we explore the dynamics of evolution when at \( t = 0 \) we induce a sudden change from a stationary shape to a non-stationary position for the inviscid or viscous drop. To reduce the infinite number of initial configurations we restrict ourselves to numerical simulations of the evolution of drops with shapes located at the stable or unstable stationary branch of the lobe. At time \( t = 0 \) we change a physical parameter thus rendering the dynamics evolving from a position that is not a stationary shape. For a given fluid system at isothermal conditions, the fluid properties as drop volume, viscosity ratio or interfacial tension are assumed constant. Hence, changing the physical parameters translates to, simply, increasing or reducing the flow intensity \( G \). However, \( G \) appears in all dimensionless parameters employed so far, i.e., \( f_1, f_2 \) and \( h \). Hence, while for experiments with
an inviscid drop, where \( f_1 \) is the sole parameter, changing \( G \) translates directly to changing \( f_1 \), in the case of a viscous drop changing \( G \) affects both \( f_2 \) and \( h \) (except for the case of creeping flow where \( h \) is zero). Thus, for the latter cases, we redefine our parameters space replacing the set \( f_1 \) and \( f_2 \) with the set \( m \) and \( f_2 \) in the dynamic Eq. (6) and in the relations given by Eq. (11), where

\[
\begin{align*}
    m &= h f_2 = \frac{f_1^4}{f_2^3} = \frac{Re}{Ca \lambda^{3/6}} = \frac{\rho a \sigma}{\mu^2 \lambda^{5/6}}.
\end{align*}
\]  

(13)

Note that \( m \) is independent of \( G \). In all the runs and figures the initial shapes of all evolution patterns are calculated using the previous theory (see, e.g., Favelukis et al.\(^{15}\)).

### A. Drops initially located on stable branch

#### 1. An inviscid drop (\( \lambda = 0 \))

It is already known\(^6,15\) that an inviscid drop with no inertia (\( f_1 = f_2 = 0 \)) is stable in all cases. Furthermore, as is described in Sec. III A, a drop that is initially located at some \( f_1 \) (associated with \( 2 < v < 2.54 \)) on the lower stable branch of the stationary lobe is stable to small disturbances. If the inertia intensity is suddenly changed to some \( f_1 \) in the range \( 0 < f_1 < 0.207 \), the shape will smoothly evolve to the new steady state on the lower branch. However, when the change in the inertia intensity drives the value of \( f_1 \) beyond 0.207, a different dynamics evolves. This experiment follows the suggestion by Sherwood\(^9\) and the results are depicted in Fig. 3(a). Here, we start with an inviscid drop located at the stationary stable branch (\( v = 2.3, f_1 = 0.200 \)). At \( t = 0 \), the inertia

![Image](image_url)

**FIG. 3.** The evolution of a drop experiencing a sudden change in the flow intensity. The drop is originally located at the stable branch. (a) \( \lambda = 0, h = \infty \), at \( t < 0 \): \( v = 2.3, f_1 = 0.200 \); at \( t > 0 \): \( f_1 = 0.22 \); (b) \( \lambda \neq 0, h = 0, v = 2.3, f_2 = 0.147 \), at \( t > 0, f_2 = 0.15 \).
strength of the flow is suddenly increased to $f_1 = 0.22$, which is greater than the critical point ($f_1 = 0.207$). As can be seen in Fig. 3(a), the drop breaks up by a mechanism of center pinching. The numerical results show also that the half-length and the uniform drop pressure do not change much during most of the evolution period, except very close to the breakup time.

2. A viscous drop in creeping flow ($\lambda \neq 0, f_1 = 0$)

In the case of a viscous drop the findings are similar to those obtained in Sec. IV A, though slightly more complex since there are more parameters and more modes of drop breakup. We start with the experiment similar to the one suggested by Sherwood\(^9\) with a viscous drop under creeping flow conditions ($\lambda \neq 0, f_1 = h = 0$). Fig. 3(b) shows a viscous drop originally located at the lower stable branch ($\nu = 2.3, f_2 = 0.147$) just before the critical point ($\nu = 2.4, f_2 = 0.148$), see Table I. At $t = 0$, the viscous strength of the flow is suddenly increased to $f_2 = 0.15$. The drop breaks in a mechanism of indefinite elongation. Other runs suggests that, the breakup dynamics is slower with a small jump in $f_2$ than with a large jump in $f_2$. It should be noted that the breakup mechanism by indefinite elongation, with pointed ends and no tip streaming observed, was also obtained experimentally by Bentley and Leal\(^16\) for slender drops extending in a slowly augmenting two-dimensional flow at near creeping flow conditions ($Re \sim 10^{-3}$-$10^{-4}$, $h \sim 0.3$). In that case the critical capillary numbers are all smaller than unity while the condition for the existence of slender drops in the axisymmetric extensional flow is $Ca^3 \gg 1$. Nevertheless, the qualitative agreement between the breakup mechanisms in the two cases is evident.

B. Drops initially located on an unstable branch

1. An inviscid drop ($\lambda = 0$)

Another set of experiments involves drops having initially unstable stationary lengths. We have already established that a drop located on the upper unstable branch at some chosen $f_1$ with $\nu > 2.54$ (and similarly on the unstable branches of all the lobes) becomes unstable to small disturbances and its shape evolves to a breakup via a center pinching mode. We shall see that a similar dynamics is also obtained whenever, at $t = 0$, the parameter $f_1$ is increased to any other larger value rendering the drop shape non-stationary. Thus, we find that a sudden arbitrary increase in the inertia intensity leads the drop shape to evolve and break up in a center pinching mode. However, when at $t = 0$ $f_1$ is decreased to any value, again rendering the drop shape non-stationary, a different evolution dynamics is observed. The drop shape evolves to the steady shape associated with $f_1$ on the lower stable branch of the first lobe and the drop does not break. Hence, any arbitrary reduction of the flow intensity, although perturbing a drop located in an unstable region, does not lead to a breakup. These patterns of dynamics (breakup for increase in $f_1$ and no breakup for a decrease in $f_1$) were observed starting at unstable branches of higher lobes as well.

Examples of such dynamics are shown in Fig. 4 where, before $t = 0$, the drop is located at $\nu = 3.0, f_1 = 0.196$, and $x_L = 39.6$. In Fig. 4(a) no change is induced; the drop shape in the upper unstable branch evolves with time until breakup occurs in a mechanism of center pinching (as described in Sec. III A). Fig. 4(b) corresponds to a sudden increase in the inertia strength of the flow (at $t = 0, f_1 = 0.200$), to a value still lesser than the critical point ($f_1 = 0.207$). The drop breaks up by a center pinching mechanism. When a small reduction in the inertia intensity is performed (depicted in Fig. 4(c)), where at $t = 0, f_1 = 0.19$, the unstable drop, originally located in the upper unstable branch, evolves smoothly to the stable branch. A similar experiment is shown in Fig. 4(d), with an even bigger backward step (at $t = 0, f_1 = 0.10$), resulting in a much faster evolution to the lower stable branch.

These modes of evolution dynamics are summarized and mapped in Fig. 5 where the phase plane of $f_1$ and $\nu$ for the first lobe is depicted. The plot shows the stable and unstable branches of the lobe in terms of $f_1$ and $\nu$, the former associated with the flow intensity and the latter related to the drop internal pressure. Note that $\nu$ is used as a coordinate in this plot since among the physical variables and parameters, $x_L, f_1$ and $P$, the latter is the only one increasing monotonically along the lobe while the former ones are multivalued. Solid arrows indicate the direction and interval of the
jump while the dashed arrows show, schematically, the trajectory to the final shape. Jumps into the white zone result in a breakup via center pinching.

2. A viscous drop ($\lambda \neq 0$)

We turn to an experiment in the absence of inertia ($h = 0$), similar to the one carried out for the inviscid drop ($h \to \infty$), and it is summarized in Fig. 5. Here we set the initial shape of the
viscous drop on the unstable branch of the single lobe, with $\nu > \nu_{cr} = 2.4$. If at $t = 0$ the flow rate is either decreased or increased (i.e., $f_2$ is initially either lower or higher but still below the critical $f_2 = 0.148$), the drop shape either evolves to the steady one on the lower branch or breaks up, respectively. In contrast to the case of inviscid drops, in this case the breakup mechanism is via indefinite elongation. The various patterns of evolution for the viscous drop in flow without inertia are depicted in Fig. 6. As in Fig. 5, solid arrows show the direction and interval of the jump while the dashed arrows indicate the trajectory to the final shape. Jumps into the white zone result in a breakup of the viscous drop via indefinite elongation.

Next, we perform evolution experiments for viscous drops with inertia effect ($h \neq 0$) in the first lobe similar to those described in Sec. IV A. Here, as explained above, we employ the set of parameters $m$ and $f_2$ and, hence, a change in the flow intensity affects only $f_2$ (and $h$) while $m$ remains unchanged. Below, are examples of various runs that were made to establish the entire behavior of the unsteady evolution dynamics. The first case is at $m = 1.37$, where the unstable stationary shape at $t < 0$ is akin to $f_2 = 0.137$ and $h = 10$ (at $\nu = 3$). When, at $t = 0$, $f_2$ is decreased, the shape evolves smoothly to that in the lower stable branch. However, when at $t = 0$, $f_2$ is augmented to 0.140 at which, with constant $m$, $h = 9.77$ (which is still below the critical one at $f_2 = 0.146$, see Table I), breakup occurs and the mechanism is by indefinite elongation, see Fig. 7(a). A similar experiment is performed for a higher level of inertia at $m = 12.9$, where the unstable stationary shape at $t < 0$ and for $\nu = 3$ (upper branch), is at $f_2 = 0.129$ ($h = 100$). Once again, when, at $t = 0$, $f_2$ is decreased, the shape, as already expected, evolves smoothly to that in the lower stable branch. However, Fig. 7(b) shows a case when at $t = 0$, $f_2$ is augmented to 0.135 at which, with constant $m$, $h = 95.6$ (which is still below the critical one at $f_2 = 0.137$, see Table I), breakup occurs. The mechanism that starts as elongation eventually evolves into a pinching at the center when a negative curvature develops there.

To augment these experiments we also studied the evolution of shapes that start at $\nu = 5$ at which, in the presence of any nonzero level of inertia, there are no stationary solutions for the shape as the point is located between the first and second lobes. The initial shape was adopted from the case of zero inertia (where at $\nu = 5$, $f_2 = 0.0993$) and four runs were made. When, at $t = 0$, $h$ was set at 10 or 100 leaving $f_2$ at 0.0993 the drop breaks up by indefinite elongation and by center pinching, respectively. However, when at $t = 0$ with $h$ set at 10 or 100, $f_2$ is reduced to 0.05 the drop evolves smoothly to the respective steady stationary shape on the stable branch of the first lobe.

FIG. 6. Phase plane showing the evolution of a viscous drop in the absence of inertia ($\lambda \neq 0, h = 0$). The shape is initially located on the stationary curve and the dynamics is due to sudden increase or decrease in the viscous flow intensity. Solid and dashed arrows indicate the direction of initial jump and resulting evolution, respectively.
FIG. 7. The evolution of a viscous drop with inertia effect, experiencing a sudden change in viscous flow intensity. The drop is originally located at the unstable branch. (a) $m = 1.37$, at $t < 0$: $v = 3.0, f_2 = 0.137$, at $t > 0$: $f_2 = 0.140$; (b) $m = 12.9$, at $t < 0$: $v = 3.0, f_2 = 0.129$, at $t > 0$: $f_2 = 0.135$.

The evolution dynamics of viscous drops in the presence of inertia are summarized in the phase planes that are shown in Fig. 8. Two cases are depicted. $m = 10$ and $m = 5$. Note that these phase planes are different from those associated with the cases $m \to \infty$ ($h \to \infty$) and $m = 0$ ($h = 0$), i.e., the cases of an inviscid drop with inertia and a viscous drop in creeping flow, respectively. Recall that for an inviscid drop, the breakup mechanism is by center pinching only (see Fig. 5) and for a viscous drop in creeping flow, by indefinite elongation (see Fig. 6). In contrast, as is depicted Figure 8, when the physical properties of the fluids are such that $m$ assumes an intermediate value, $0 < m < \infty$, the two breakup mechanisms can occur. In Fig. 8(a) we show a case with a relatively high value of $m$. Indeed, if the jump in $f_2$ is not too high and the dynamics starts close to the stationary curve, breakup mechanism is by center pinching, as is expected for a high $h$. However, when the jump in $f_2$ is larger, since $h = mf_2$ and with $m$ being constant, $h$ decreases to a level at which, in that region, the breakup mechanism changes to indefinite elongation. When $m$ is initially low, see Fig. 8(b), the region of breakup via indefinite elongation is the dominant along most of the stationary curve. However, at $v$ values close to 4 where $f_2$ assumes very low values the corresponding $h$ has high enough values and a sub-region of breakup mechanism by center pinching still exists.

We conclude this discussion by recognizing that, perhaps, the most significant point of transition between the breakup mechanisms is when it coincides with the point at which the drop first loses its stability, i.e., at $v = v_{cr}$. This point is of course the bifurcation point between the stable and unstable branches of the first lobe at which $f_2$ is maximum and the rate of increase of $x_L$ is infinite.
The transition, at this point, is obtained for a system with $m = 6.4$ at $v = 2.42$ ($h$ and $f_2$ being 45 and 0.142, respectively).

V. CONCLUSIONS

The dynamic evolution of a slender drop in an extensional flow was studied numerically taking into account weak inertia of the ambient flow. Stationary shapes were chosen for the initial conditions when the flow intensity was kept constant. When the flow was subjected to a sudden change the initial shape was no longer stationary. It was shown that the dynamics of the shape evolution can lead to a breakup of the drop or to a stable stationary shape. Whenever the evolution starts at a stable stationary shape, and the initial flow intensity remains below the critical value at the bifurcation point, the drop will evolve into a steady stable shape. On the other hand, if the initial flow intensity is above the critical one, breakup will occur. Whenever the evolution starts at an unstable stationary
shape at the initial or slightly higher flow intensity the drop breaks up. However, if the flow intensity is reduced, even slightly, the drop will evolve to the respective steady shape on the lower stable branch. When the drop breaks up two mechanisms were realized. At low inertia intensity (low \( h \)) the drop elongates indefinitely. At higher inertia intensity (high \( h \)) the drop breaks up into two equal parts via a center pinching mechanism. However, in all intermediate cases, where \( 0 < m < \infty \), the two mechanisms are realized. It should be added that while most of the evolution dynamics was studied from positions on the first lobe, the few experiments performed starting at the second lobe and in between, yielded similar results, i.e., evolution to the lowest (stable) branch when the inertia intensity slightly decreases and breakup of the shape when the intensity remains untouched or slightly increases, suggesting that no other results are expected. Of course, in theory, if the initial location is at a much higher lobe and the resulting parts of the first breakup are still in an unstable region, they are expected to further disintegrate and a greater number of daughter may be produced. Maps of the first lobes summarizing the various cases of dynamics are depicted in the figures for various cases of the relative inertia intensity.

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