Iterative Linearization Scheme for Convex Nonlinear Equations: Application to Optimal Operation of Water Distribution Systems

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Abstract: Convex equations exist in different fields of research. As an example are the Hazen-Williams or Darcy-Weisbach head-loss formulas and chlorine decay in water supply systems. Pure linear programming (LP) cannot be directly applied to these equations and heuristic techniques must be used. This study presents a methodology for linearization of increasing or decreasing convex nonlinear equations and their incorporation into LP optimization models. The algorithm is demonstrated on the Hazen-Williams head-loss equation combined with a LP optimal operation water supply model. The Hazen-Williams equation is linearized between two points along the nonlinear flow curve. The first point is a fixed point optimally located in the expected flow domain according to maximum flow rate expected in the pipe (estimated through maximum flow velocities and pipe diameter). The second point is the calculated flow rate in the pipe resulting from the previous iteration step solution. In each iteration step, the linear coefficients are altered according to the previous step’s flow rate result and the fixed point. The solution gradually converges closer to the nonlinear head-loss equation results. The iterative process stops once both an optimal solution is attained and a satisfactory approximation is received. The methodology is demonstrated using simple and complex example applications. DOI: 10.1061/(ASCE)WR.1943-5452.0000275. © 2013 American Society of Civil Engineers.

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Introduction

Overcoming nonlinearity properties of nonlinear optimization models through linearization is a well-known challenge. This study presents a methodology to cope with this difficulty in case the objective function and/or constraints are convex.

An example of such a problem is optimizing the operation of water distribution systems (WDSs) in which pump schedules are searched for minimizing operational costs while maintaining flow, pressure, and tank water levels. The head-loss relationship between flow and head (i.e., the Hazen-Williams (HW) or Darcy-Weisbach formulas) create nonlinear convex equations, thus forming a nonlinear optimization model. If only the head-loss equation would hold a linear relationship between flow and head, then the optimal operation problem could have been cast in a linear programming (LP) framework and efficiently solved. Because flows in pipes are unknown, linearization is problematic because the linearization domains are unknown. This study suggests an overall iterative linear programming scheme for dealing with these difficulties.

The rest of the paper [extending Price and Ostfeld (2011)] is organized as follows: literature review on optimal operation of water distribution systems, description of the proposed methodology, and demonstration of its capabilities on two example applications.

Literature Review

Subsequent to the well-known least-cost design problem of water distribution systems (Karmeli et al. 1968; Jacoby 1968; Alperovits and Shamir 1977), optimal operation is probably the most explored topic in water distribution systems management. Since 1970, a variety of methods were developed to address this problem, including the utilizations of dynamic programming (DP), linear programming, predictive control, mixed-integer nonlinear programming (MINLP), metamodeling, heuristics, and evolutionary computation. Ormsbee and Lansey (1994) classified time optimal water distribution systems control models through systems type, hydraulics, and solution methods. This section reviews the current literature on this subject.

Dynamic Programming

Dreizin (1970) was the first to suggest an optimization model for water distribution systems operation through a DP scheme coupled with hydraulic simulations for optimizing pumping scheduling of a regional water supply system supplied by three pumping units. Sterling and Coulbeck (1975) used a dynamic modeling approach to minimize the costs of pump operation of a simple water supply system. Carpenter and Cohen (1984) developed a decomposition-coordination methodology for partitioning a water supply system into small subsystems that could be solved separately (i.e., decomposed) using dynamic programming, and then merged (i.e., coordinated) at the final solution. Houghtalen and Loftis (1989) suggested aggregating training simulations with human operational...
knowledge and dynamic programming to minimize operational costs. Ormsbee et al. (1989) developed a coupled dynamic programming and enumeration scheme for a single pressure zone in which the optimal tank trajectory is found using dynamic programming and the pump scheduling using enumeration. Zessler and Shamir (1989) used an iterative dynamic programming method to find the optimal scheduling of pumps of a regional water supply system. Lansey and Awunah (1994) used a two-level approach in which the hydraulics and cost functions of the system are generated first off-line followed by a dynamic programming model for pump scheduling. Nitivattananon et al. (1996) utilized heuristic rules combined with progressive optimality to solve a dynamic programming model for optimal pump scheduling. McCormick and Powell (2003) utilized a stochastic dynamic program framework for optimal pump scheduling in which daily demand for water is modeled as a Markov process.

**Linear Programming**

Olshansky and Gal (1988) developed a two-level linear programming methodology in which the distribution system is partitioned into subsystems for which hydraulic simulations are run and serve further as parameters in an LP model for pumps optimal scheduling. This approach was used also by Jowitt and Germanopoulos (1992), who developed a linear programming model to optimize pump scheduling in which the LP parameters are set through off-line extended period hydraulic simulation runs. Diba et al. (1995) used graph theory coupled with a linear programming scheme for optimizing the operation and planning of a water distribution system including reliability constraints.

**Predictive Control**

Coulbeck (1980) and Coulbeck et al. (1988a, b) suggested hierarchical control optimization frameworks for optimal pump operation. Biscos et al. (2002) used a predictive control framework coupled with MINLP for minimizing the costs of pump operation. Biscos et al. (2003) extended Biscos et al. (2002) to include the minimization of chlorine dosage.

**Mixed Integer**

Ulanicki et al. (2007) developed a mixed-integer model for tracking the optimal reservoir trajectories based on the results of an initially relaxed continuous problem. Pulido-Calvo and Gutiérrez-Estrada (2011) presented a model for both sizing storage and optimizing pump operation utilizing a framework based on a MINLP algorithm and a data-driven (neural networks) scheme.

**Nonlinear Programming**

Chase (1990) used an optimization—simulation framework coupling the general reduced gradient GRG2 (Lasdon and Waren 1986) with a water distribution system simulation model WADISO (Gessler and Walski 1985) for minimizing pump cost operation. Brion and Mays (1991) developed an optimal control simulation—optimization framework for minimizing pump operation costs in which the simulation solves the hydraulic equations and the optimization utilizes the nonlinear augmented Lagrangian method (Brion 1990). Pezeshk et al. (1994) linked hydraulic simulations with nonlinear optimization to minimize the operation costs of a water distribution network. Cohen et al. (2000a, b, c) presented three companion papers on optimal operation of water distribution systems using nonlinear programming: with water quality considerations only (Cohen et al. 2000a), with both flow and quality (Cohen et al. 2000b), and with both flow and quality (Cohen et al. 2000c).

**Metamodeling**

Broad et al. (2005) used an artificial neural network (ANN) as a metamodel for optimizing the operation of a water distribution system under residual chlorine constraints. Shamir and Salomons (2008) developed a framework for real-time optimal operation integrating an aggregated/reduced model, an artificial neural network, and a genetic algorithm (GA). Broad et al. (2010) extended Broad et al. (2005) through comparing four different metamodeling scenarios and suggesting skeletonization procedures.

**Heuristics**


**Evolutionary Computation**


**Commercial Modeling Tool**

Commercial applications for energy minimization have been developed by companies such as Derceto (http://www.derceto.com), Bentley (http://www.bentley.com/en-US/Solutions/Water+and+Wastewater), and MWH Soft (http://www.mwhglobal.com). These applications allow system design, optimal pump scheduling, and system operation while minimizing system operation cost and optimizing water supply.

**Methodology**

The model developed in this study is based on a general algebraic modeling system/coin-or branch and cut (GAMS/CBC) ILP (http://projects.coin-or.org/Cbc) minimal-cost optimal-operation WDS model developed and actively used in Tahal Consulting Engineers Ltd., Israel. The model determines the optimal operation time for each pumping unit combination in each pumping station for each hour of the simulation period. The model determines hourly flows between nodes to supply hourly demand constraints at demand.
nodes and maximize the utilization of water tank/reservoir volumes constrained by water balance closure over the optimization period. The methodology is based on a water balance model with no hydraulic equations (i.e., no head-loss equations or head at nodes). Manually calculated maximum or minimum flow rate constraints between nodes may be entered. The optimization period is annual (365 days × 24 h) or weekly (7 days for each of the 12 months × 24 h).

### Linearization of the Hazen-Williams Head-Loss Equation

Extending the water balance model described previously to include hydranetics, the nonlinear HW head-loss equation, shown in Eq. (1), is partitioned into two subequations. Eq. (2) represents the constant part of the HW equation dependent only on pipe geometry. Eq. (3) describes the linearization of the nonlinear flow $Q_{t,ij}$ as a linear equation, subject to the coefficients $A$ and $B$. Eq. (4) is the combined linear head-loss equation.

$$dH_{t,ij} = 1.131 \times Q_{t,ij}^{1.852} \times HC_{t,ij}^{-1.852} \times D_{l,ij}^{-4.87} \times 10^6 \times L_{t,ij} \quad \forall i, j \in N, t \in T$$

where $t =$ time index (hours); $T =$ group of all time indexes (1, 7, 365); $i, j =$ index of pipe origin and destination nodes, respectively; $N =$ group of all node indexes; $dH_{t,ij} =$ head loss in pipe variable (m); $Q_{t,ij} =$ flow rate in pipe variable (m$^3$/h); $HC_{t,ij} =$ Hazen-Williams head-loss coefficient constant (-); $D_{t,ij} =$ pipe diameter constant (mm); and $L_{t,ij} =$ pipe length constant (m).

$$R_{t,ij} = 1.131 \times HC_{t,ij}^{-1.852} \times D_{ij}^{-4.87} \times 10^6 \times L_{t,ij} \quad \forall i, j \in N$$

where $R_{t,ij} =$ pipe resistance’s coefficient constant.

$$Q_{t,ij}^{1.852} = A_{t,ij} \times Q_{t,ij}^t + B_{t,ij}^t \quad \forall i, j \in N, t \in T$$

where $A_{t,ij}, B_{t,ij}^t =$ linear equation coefficients constants.

$$dH_{t,ij} = (A_{t,ij}^t + Q_{t,ij}^t + B_{t,ij}^t) \times R_{t,ij} \quad \forall i, j \in N, t \in T$$

### Determining the $A$, $B$, and $R$ Coefficients

The pipe resistance matrix $R$ is precalculated using Eq. (2) before the first iteration step as constants. The linear equation’s $A$ and $B$ coefficients are calculated before each iteration step as constants using one of the following two methods.

#### Method A

Method A creates an initial rough linearization of the $Q_{t,ij}^{1.852}$ curve through the origin and $Q_{t,ij}^{max}$ using Eq. (5). A maximum expected flow rate $v$ in all the pipes is set. $Q_{t,ij}^{max}$ is the maximum expected flow rate in a pipe; for $v$ and the pipe’s diameter, $Q_{t,ij}^{max}$ is not an upper bound constraint. In the examples that follow, a value of $v = 2$ m$^3$/h was used as the high range of normal flow rates expected in a pipe for standard pipe sizing. The $A_{t,ij}$ and $B_{t,ij}^t$ coefficients are found for a linear line intersecting two points: the origin [0, 0] and the point $[\alpha Q_{t,ij}^{max} (\alpha Q_{t,ij}^{max})^{1.852}]$, as described in Fig. 1 (line A). The value of $\alpha = 0.7$ was optimally found to best represent the previous domain linearly while minimizing accumulative error between the linear line and the $Q_{t,ij}^{1.852}$ curve.

#### Method B

Method B iteratively creates a linearization of the $Q_{t,ij}^{1.852}$ curve between the two points $(Q_{t,ij}^{1.852})_1$ and $(Q_{t,ij}^{1.852})_2$ using Eqs. (6) and (7), as shown in Fig. 1 (line C). $(Q_{t,ij}^{1.852})_1$ is the resulting flow rate derived from the previous iteration step, and $(Q_{t,ij}^{1.852})_2$ is a fixed flow rate point for

![Fig. 1. Linearization of $Q_{t,ij}^{1.852}$ for different cases (pipe diameter 250 mm)](image-url)

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per pipe, which is determined relative to \( Q_{i,j}^{\text{max}} \). \( Q_{i,j}^{\text{fix}} \) serves as a constant anchor to which the linearization is made throughout all the iteration steps. The anchor maintains a positive minimal slope value for the linear line when a value of \( (Q_{i,j}^{\text{fix}})_r \) is close to zero \( (\mid Q_{i,j}^{\text{fix}} \mid \ll Q_{i,j}^{\text{fix}}) \) and steepens the slope for high \( \mid Q_{i,j}^{\text{fix}} \mid \gg Q_{i,j}^{\text{fix}} \). The \( Q_{i,j}^{\text{fix}} \) point was found to be optimally located at \( Q_{i,j}^{\text{fix}} = \beta Q_{i,j}^{\text{max}} \). The value of \( \beta = 0.44 \) was optimally found for minimal accumulative error between the \( O_{i,j}^{1.852} \) curve and the two linear lines connecting \( Q_{i,j}^{\text{fix}} \) and \( Q_{i,j}^{\text{max}} \) and connecting \( Q_{i,j}^{\text{max}} \) and the origin, as shown in Fig. 1 (line B). If the linearization of the \( O_{i,j}^{1.852} \) curve were to be made directly between \( Q_r \) and the origin (not using \( Q_{i,j}^{\text{fix}} \)), then for \( Q_r \) values that are close to zero, the slope of the linear line approaches perpendicular with the x-axis \( (A_{i,j} \to 0) \), making it near impossible, in the following iteration step, for the model to increase the \( (\mid Q_{i,j}^{\text{fix}} \mid) \) flow rate (to gain head loss between nodes) as the relation between flow rate \( (\mid Q_{i,j}^{\text{fix}} \mid) \) and head loss \( (dH_{i,j}) \), given in Eq. (4), becomes negligible.

Method A: Calculating A, B coefficients:

\[
A_{i,j} = \frac{(\alpha Q_{i,j}^{\text{max}})^{1.852}}{\alpha Q_{i,j}^{\text{max}}}; \quad B_{i,j} = 0 \quad \forall i, j \in N, t \in T \quad (5)
\]

where \( \alpha = 0.7 \) (\( \cdot \)); and \( Q_{i,j}^{\text{max}} \) = maximum expected flow rate constant \( (m^3/h) \).

Method B: Calculating A, B coefficients:

\[
A_{i,j} = \frac{(\mid Q_{i,j}^{\text{fix}} \mid)^{1.852} - (\mid Q_{i,j}^{\text{fix}} \mid)^{1.852}}{(\mid Q_{i,j}^{\text{fix}} \mid)^{1.852} + (\mid Q_{i,j}^{\text{fix}} \mid)}; \quad B_{i,j} = 0 \quad \forall i, j \in N, t \in T \quad (6)
\]

where \( Q_{i,j}^{\text{fix}} \) = anchor flow rate constant \( (m^3/h) \); and \( \gamma \) = constant to prevent division by zero.

\[
B_{i,j} = (\mid Q_{i,j}^{\text{fix}} \mid) \times ((\mid Q_{i,j}^{\text{fix}} \mid)^{1.852} - A_{i,j}) \quad \forall i, j \in N, t \in T \quad (7)
\]

**Head-Loss Linearization Algorithm**

The proposed algorithm includes the stages described in this section (see Fig. 2). The model used is described subsequently in this paper.

**General Discussion**

Method A is used in the first iteration step in order to generate starting conditions (flow rates) for the second iteration step, which uses Method B. The following iteration steps all use Method B to calculate the A and B linear coefficients connecting between the fixed point along the convex curve and the previous iteration step’s resulting flow rates. The advantage of Method A is that the linearization passes through the origin, allowing the head loss to be correctly calculated even if the flow rates are positive or negative. Method A is a less accurate general approximation of the convex curve, while Method B is a more accurate representation of the curve, mainly in the vicinity of the fixed point and the previous step’s flow rate.

In the early stages of the development of the algorithm, an attempt was made to initially set the A and B coefficients of each pipe and each hour in the first iteration step. In following iteration steps, the coefficients were gradually modified toward convergence using fixed step changes in each iteration step. This method returned extremely long solution times due to the fact that in some pipes the flow rates changed dramatically between the first and final iteration steps. Many iteration steps were necessary to allow the change of the coefficients following the change in flow rates over the iteration steps. The recalculating of the A and B coefficients in each iteration step as proposed in the method used returns shorter solution times because of the need for fewer iteration steps and the fact that the system has the freedom to radically change flow rates between iteration steps followed by coefficient changes.

**Stage I—LP Model**

This stage acts as a filter to whether the problem is at all feasible (see Fig. 2). The model is solved using a water balance LP model with no hydraulic constraints. If the problem is infeasible, the algorithm stops and the user is flagged that the problem is infeasible due to a water balance problem. If an optimal solution is found, the algorithm proceeds to Stage II.

**Stage II—Initial Linearization (Using Method A)**

This stage generates optimal rough hydraulic-based flow rates that will serve as starting conditions for Stage III (see Fig. 2). The A, B, and R coefficients are calculated using Method A. A LP water balance combined hydraulic constrained model is solved. The resulting flow matrix \( (Q_{i,j}^1) \) is passed to the next stage. If the problem is infeasible, the algorithm stops and the user is flagged. The resulting flow rates from Stage II may differ significantly from the flow rates resulting from Stage I due to the hydraulic constraints. Therefore, the resulting flow rates from Stage II make preferable starting conditions for Stage III.

**Stage III—Iterative Linearization (Using Method B)**

The flow rate matrix \( (Q_{i,j}^1) \) found in Stage II serves as the \( Q_r \) matrix for the first iteration step in the current stage (see Fig. 2). The \( A_{i,j} \) and \( B_{i,j} \) coefficients are calculated according to Method B. A LP water balance combined hydraulic constrained model is solved, resulting in a new matrix of flow rates \( (Q_{i,j}^1) \). Using Eq. (8), the maximum error between the head losses calculated using Eqs. (1) and (4) is found for each pipe for each hour. While maxErr > 0, the iterative procedure repeats (i.e., the \( A_{i,j} \) and \( B_{i,j} \) coefficients are recalculated using Method B on the latest \( (Q_{i,j}^1) \) flow rate matrix and the model is rerun). The process successfully stops when maxErr = 0 (exactly equals zero) or fails if a maximum number of iteration steps is reached.

Calculating HW linearization maximum error:

\[
\text{maxErr} = \max \left( \frac{\mid (Q_{i,j}^1)^{1.852} - (A_{i,j} (Q_{i,j}^1)_r + B_{i,j}) \mid}{\mid (Q_{i,j}^1)^{1.852} \times 100} \right) \quad \forall i, j \in N, t \in T \quad (8)
\]

where maxErr = maximum error between the \( Q_r^{1.852} \) and \( (AQ_r + B) \) variables (%).

**Piecewise Linearization and Successive Linear Programming versus the Proposed Algorithm**

Berghouse and Kuczera (1997) suggested approximating the nonlinear head loss flow relationship in a pipe using a piecewise linear function. To allow flow reversal during a simulation, each pipe was duplicated. This substantially increased the size of the linear
programming formulation and was found to be less efficient than other simulation algorithms [e.g., Todini and Pilati (1988)].

Exploring a piecewise linear programming approach for pump optimal operation has not been investigated in the literature and could be further developed. However, such a formulation will increase substantially the linear programming problem size and could not accommodate for maximum pressure constraints [see Eq. (15)]. The reason for not being able to accommodate maximum pressure constraints resides in the theory of convex programming in which it can be shown (Hiller and Lieberman 2001) that a nonlinear optimization problem can be equivalently approximated through piecewise linearization if the objective function is convex, all constraints are convex, and all are of a less-equal type. Inclusion of the maximum pressure/head restrictions results in convex greater-equal constraints, even if the constraints are multiplied by \(-1\) to become less-than constraints; relative to the head loss convex, the constraint remains greater than because both sides of the equation are multiplied by \(-1\).

The water system in this study is modeled as a directed network. Some pipes are defined as having a one-directional flow \(0 \leq Q_{ij}\) and other pipes allow bidirectional flow \((-\infty < Q_{ij} < \infty)\). In previous research (Berghoue and Kuczera 1997), the piecewise approach was considered for the linearization of the convex \(Q^{1.852}\) curve. The main disadvantage of piecewise linearization is that the flow rates must be greater than zero and the convex function must be positively increasing. In Stage II [Eq. (5)], the linearization is made smoothly through the origin, allowing the model to pass from a positive to negative flow direction. If in Stage III a flow direction in a pipe changes from positive to negative, this will cause a negative head loss over the pipe. The negative head loss is corrected

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**Legend**

- \(\text{maxErr}\) – Maximum difference between HW and \((AQ+B)xR\) head loss.
- \(\text{pr}\) – Previous iteration step’s annual operational cost.
- \(\text{cu}\) – Current iteration step’s annual operational cost.
- \(\text{FP}\) – Flow penalty flag. If true flow change penalty is enforced to minimize flow changes relative to previous step’s flow results.

**Fig. 2. Head loss linearization algorithm**
in the following iteration stage [Eqs. (6) and (7)]. Further, the proposed algorithm may be applied to decreasing concave functions such as pump curves (not demonstrated in this paper), and the successive linear iterations in this study do not require the computation of gradients as a typical successive LP (SLP) scheme necessitates.

Preventing Solution Oscillation

In some model conditions, after several Stage III iteration steps the solution of the model converges on two similar optimal solutions and oscillates between them repeatedly in the following iteration steps. The oscillation is caused in cases in which, for instance, in a certain day a pumping station is activated for 5 h out of an 8-h low electrical tariff period. For the minimal cost part of the LP water balance model, there is no difference which of the 5 h the pumping station is activated because the electrical tariff is the same in all of the 8 h, making the annual electrical cost the same. The water balance part of the model in one iteration step may activate the pumping station consecutively during the first 5 h, while in the next iteration step the model may activate the pumping station in the last 5 h of the 8-h period. In this example, in one iteration step the flow in a pipe in the first hour is the pumping station’s flow rate, and in the next iteration the flow rate changes to zero. In the following iteration step, the flow rate is again not zero and so on. For the hydraulic part of the model, the flow rate change \( \Delta Q_{ij} \), from a certain flow rate in one iteration step to a radically different flow rate in the next iteration step does not allow the \( A'_{ij} \) and \( B'_{ij} \) coefficients to converge over a certain flow rate in the pipe per hour. This does not allow the minimization of the maxErr, and Stage III repeats until reaching a maximum number of iteration steps and fails.

To prevent solution oscillation, after each Stage III iteration, the current iteration step’s annual operating cost \( cu \) is compared to the previous iteration step’s annual cost \( pr \) using Eq. (9). If the difference between \( pr \) and \( cu \) is smaller than a defined value and the current solution is smaller than the previous solution, the FP flag (initially false) is made true (see Fig. 2). Once the FP flag is true, it will remain true until the algorithm ends.

Once the FP flag is made true, a penalty constraint is enforced in the model to minimize flow rate changes in the pipes per hour relative to the flow rates in the previous iteration step. This method eliminates the oscillation effect and dramatically reduces the number of iteration steps needed for the optimal solution with maxErr = 0.

Conditions for preventing Stage III solution oscillation:

\[
\text{if } \left( cu = 0 \text{ or } \frac{pr - cu}{cu} \leq 0.1\% \text{ and } cu < pr \right) \text{ then FP = true}
\]  

(9)

where \( pr \) = annual cost result from previous iteration step [in new Israeli shekels (NIS)]; \( cu \) = annual cost result from current iteration step (NIS); and FP = flag to enforce a flow change penalty constraint (true or false).

Modeling System and Solver

The algorithm is utilized in a dedicated application entitled FLOWS written in MS-C# 2010. The FLOWS application includes a graphics user interface that enabled defining the model, applying the algorithm, and displaying the optimal solution results. The FLOWS application activates the GAMS (http://www.gams.com) application to solve .gms files constructed by the application. In each iteration step, the FLOWS application builds a .gms file and calls GAMS to solve the model with the COIN-OR/CBC (coin-or branch and cut) solver (https://projects.coin-or.org/Cbc). The solver results are output from the GAMS application into text files. The text files are read into the FLOWS application and processed according to algorithm stage I, II, or III to create a new .gms file for the next iteration step.

Originally the model was solved using GAMS/MINOS. The COIN-OR/CBC solver was selected because it returns significantly shorter solution times relative to the GAMS/MINOS. This may be explained by the fact that a specialized current LP code should always outperform a general purpose nonlinear programming (NLP) code on a LP problem.

Solution times are for using a Lenovo T61, T7100@1.80 GHz, 0.97 GB of RAM. GAMS solution time of a single .gms file is approximately 2 to 15 s depending on the water distribution system size, complexity, and simulation period. Each iteration step includes 100 iterations, which takes 10 s to minutes to create and read the .gms file and the resulting text files. The optimal solution is achieved within two to six iteration steps, with a total solution time varying between several seconds for a small and simple model to several minutes for a large and complex model.

Model Description

A simplified description of the LP model used in this study is presented subsequently. Hydraulic head loss–related constraints, variables, and constants are marked with “(*HY).” Hydraulic constraints are omitted in Stage I and included in Stages II and III. Penalty constraints, variables, and constants for minimizing flow rate changes between iteration steps are marked with “(*FP).” The penalty constraints are included in Stage III if FP = true.

Objective Function

Minimize annual operation cost and minimize flow change (*FP):

\[
\min \left( \sum_{p=1}^{P} \sum_{t=1}^{T} PQ_p \times WP_p \times PE \times Tar^2_p \right) \\
+ \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} Pos_{ij} + Pneg_{ij} \times 100 \right) 
\]  

(10)

where \( p \) = pump station index; \( P \) = group of all pump nodes; \( PQ_p \) = pump nominal flow rate constant (m\(^3\)/h); \( WP_p \) = pump variable speed change variable (-) \([0 \leq WP_p \leq 1]\); \( PE \) = pump nominal energy consumption constant (kWh/m\(^3\)); \( Tar^2_p \) = hourly tariff charge (NIS/kWh); \( Pos_{ij} \) = positive penalty flow rate change per pipe per hour variable (m\(^3\)/h) \([Pos_{ij} \geq 0]\) (*FP); and \( Pneg_{ij} \) = negative penalty flow rate change per pipe per hour variable (m\(^3\)/h) \([Pneg_{ij} \geq 0]\) (*FP).

The use of a constant value for nominal energy consumption is highly inaccurate but this approximation was made because the purpose of this paper is to illustrate the linearization of the Hazen-Williams equation.

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Constraints

Node water balance:

\[ \sum_{i}^{N} Q'_{i,j} = \sum_{k}^{N} Q'_{j,k} \quad \forall \ t \in T, \ \forall \ j \in N \]  

(11)

Pump node output:

\[ \sum_{i}^{N} Q'_{i,p} = PO_{p} \times W'_{p} \quad \forall \ t \in T, \ \forall \ p \in P \]  

(12)

Water tank hourly and annual water balance:

\[ V'_{i} = V'^{i-1} + \sum_{i}^{N} Q'_{i,r} - \sum_{j}^{N} Q'_{r,j} \quad \forall \ t \in T, \ \forall \ r \in R; \]

\[ V'^{i-1} = V'^{i-\text{start}} + \sum_{i}^{N} Q'^{i-1} - \sum_{j}^{N} Q'^{j-1} \quad \forall \ r \in R \]  

(13)

where \( V'_{i} \) = water volume in water tank variable (m\(^3\)) \([V'^{\text{min}} \leq V'_{i} \leq V'^{\text{max}}]\); and \( V'^{\text{max}}, V'^{\text{min}} \) = water tank water volume maximum and minimum constants, respectively (m\(^3\)).

Head change at water tank relative to water volume in water tank (*HY):

\[ H'^{i-1} \times (V'^{r} - V'^{\text{min}}) = (V'^{i} - V'^{\text{min}})(H'^{r} - H'^{\text{max}}) + H'^{r} \times (V'^{r} - V'^{\text{max}}) \quad \forall \ t \in T, \ \forall \ r \in R \]  

(14)

where \( H'^{\text{max}}, H'^{r} \) = water maximum and minimum tank head constants, respectively (m); \( H'^{i} \) = water head at water tank variable (m); \( r \) = water tank index; and \( R \) = group of all water tank nodes.

Minimum and maximum head constraints at nodes (*HY):

\[ H'^{\text{min}} \leq H'^{i} \leq H'^{\text{max}} \quad \forall \ i \in N \]  

(15)

where \( H'^{i} \) = head at the \( i \)th node (m); and \( H'^{\text{min}}, H'^{\text{max}} \) = minimum and maximum heads at the \( i \)th node (m), respectively.

Demand node water balance:

\[ \sum_{i}^{N} Q'_{i,d} = D_{d} \quad \forall \ t \in T, \ \forall \ d \in D \]  

(16)

where \( D_{d} \) = consumer demand at node constant (m\(^3\)/h); \( d \) = demand node index; and \( D \) = group of all demand at nodes.

Pump total head (*HY):

\[ TH'^{p}_{i,j} = aH'^{p}_{j} \quad \forall \ t \in T, \ \forall \ p \in P, \ \forall \ j \in N; \]

\[ aH'^{p}_{i,j} = 0 \quad \forall \ t \in T, \ \forall \ i \notin P, \ \forall \ j \in N \]  

(17)

where \( TH'^{p}_{i,j} \) = total pump head variable (m) \([0 \leq TH'^{p}_{i,j} \leq TH'^{\text{max}}] \); and \( aH'^{p}_{i,j} \) = artificial head gain or loss in pipe variable (m).

Dynamic head loss in pipes (*HY):

\[ H'^{i} - H'^{j} = (A'^{i}_{j} \times Q'^{i}_{j} + B'^{i}_{j}) \times R_{i,j} + aH'^{i}_{j} \quad \forall \ i, j \in N, t \in T \]  

(18)

Penalty for flow rate change relative to previous iteration step flow rates (*FP):

\[ Q'^{i}_{j} - (Q'^{i}_{j})_{k} = P_{o}Q'^{i}_{j} - P_{n}Q'^{i}_{j} \quad \forall \ i, j \in N, t \in T \]  

(19)

Example Applications

The model is demonstrated in two example applications of increasing complexity as outlined as follows.

**Example 1: Basic Illustrative Example Application**

The algorithm is initially demonstrated on a basic simple water distribution system (see Fig 3). Results are demonstrated for an average week in June in which the highest demand rates occur. The system includes an unlimited water source node with a head of +50 m with all elevations relative to sea level, a pumping station node with a maximum flow rate of 250 m\(^3\)/h and total water head (TH) of 80 m, a tank node with an operational volume of 2,000 m\(^3\) and a constant water level of +100 m, a junction node, and a demand node with an annual consumption of 1.0 million m\(^3\)/year and a minimal supply head requirement of +90 m. The demand node has a monthly consumption distribution as presented in Table 1. For each month, the average weekly demand is calculated by dividing the monthly demand by the number of weeks in each month. The demand node has a daily consumption distribution as presented in Table 2 and an hourly demand of 10% between 7 a.m. and 4 p.m. (10 h per day). During an average week in June, the hourly demand flow rates are between 212.52 m\(^3\)/h (Saturday) and 413.49 m\(^3\)/h (Thursday). All pipes have an internal diameter of 250 mm, a length of 1.0 km, and a Hazen-Williams coefficient of 120.

Electrical tariff charges are given in units of NIS/kWh. The tariff is grouped into three annual periods: summer (July–August), winter (December–February), and intermediate (March–June and September–November). The tariff is grouped into three daily periods: Sunday–Thursday, Friday and Saturday. According to the session group and the daily period, the hours of the day are grouped into three hourly periods: low rate (marked S), moderate rate (marked G), and peak rate (marked P).

**Case I—Water Balance**

The previous water distribution system was initially solved using the water balance minimal cost LP model with no hydraulic
The resulting minimal annual operation cost is 97,225 NIS/year (see Fig. 4). The demand hours are mostly during peak tariff hours (215 to 445 m$^3$/h), marked as output in Fig. 4. As a result, the tank is fully filled during low tariff hours and fully emptied during high tariff hours.

**Case II—Water Balance Combined Hydraulic Head-Loss Constraints**

The previous water distribution system was solved using the water balance minimal cost LP model combined with hydraulic head-loss constraints. The resulting minimal annual operation cost is 97,225 NIS/year. The behavior of the pumping station and tank are the same as in Case I (see Fig. 4). The solution was achieved within 50 s and after three Stage III iteration steps (maxErr = 0%).

The annual electrical cost is the same as in Case I because the hydraulic constraints do not change the scheduling of the pumping station. The hydraulic constraints reflect the hydraulic conditions that will occur under the current flow rates. As a result, the head at the demand node varies between +110 and +60 m. High head values at the demand node occur when the pumping station is working to fill the tank, and low head values occur when the pumping station is not working and the supply to the demand node is supplied from the water tank. Because no minimal head constraints are implemented at the demand node, the head at the demand node may be lower than the minimal requirement of +90 m.

**Case III—Water Balance Combined Hydraulic Head Loss and Minimal Head Constraints**

The system was solved using the same model as in Case II with the addition of a minimal water head constraint at the demand node of +90 m during hours when demand is not zero. Note the constraints. The resulting minimal annual operation cost is 97,225 NIS/year (see Fig. 4). The demand hours are mostly during peak tariff hours (215 to 445 m$^3$/h), marked as output in Fig. 4.

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Table 1. Monthly Demand Multipliers for All Demand Nodes

<table>
<thead>
<tr>
<th>Month</th>
<th>Multiplier (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>6.3</td>
</tr>
<tr>
<td>February</td>
<td>6.3</td>
</tr>
<tr>
<td>March</td>
<td>7.0</td>
</tr>
<tr>
<td>April</td>
<td>7.9</td>
</tr>
<tr>
<td>May</td>
<td>9.2</td>
</tr>
<tr>
<td>June</td>
<td>9.9</td>
</tr>
<tr>
<td>July</td>
<td>10.8</td>
</tr>
<tr>
<td>August</td>
<td>10.6</td>
</tr>
<tr>
<td>September</td>
<td>9.7</td>
</tr>
<tr>
<td>October</td>
<td>8.4</td>
</tr>
<tr>
<td>November</td>
<td>7.3</td>
</tr>
<tr>
<td>December</td>
<td>6.6</td>
</tr>
</tbody>
</table>

Table 2. Daily Demand Multipliers for All Demand Nodes

<table>
<thead>
<tr>
<th>Day</th>
<th>Multiplier (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sunday</td>
<td>15.7</td>
</tr>
<tr>
<td>Monday</td>
<td>14.3</td>
</tr>
<tr>
<td>Tuesday</td>
<td>12.9</td>
</tr>
<tr>
<td>Wednesday</td>
<td>13.6</td>
</tr>
<tr>
<td>Thursday</td>
<td>17.9</td>
</tr>
<tr>
<td>Friday</td>
<td>16.4</td>
</tr>
<tr>
<td>Saturday</td>
<td>9.2</td>
</tr>
</tbody>
</table>

---

Fig. 4. Basic example application—Pump station and demand node hourly flow rates, average week in June (Cases I and II)
differentiation between the water head supplied to the consumer and the water pressure supplied, which is dependent on the elevation of the node. In this example, if the elevation of the consumer is +60 m, according to the minimal head constraint the minimal pressure to be supplied will be 30 m. The resulting minimal annual operation cost is 106,191 NIS/year. The solution was achieved within 80 s and after six Stage III iteration steps (maxErr = 0%).

To maintain the minimal head constraint at the demand node, the pumping station is activated during demand hours directly to supply (see Fig. 5). Because almost all demand is supplied by the pumping station, the tank’s volume is mostly unused. The head at the demand node varies between +100 and +90 m according to the pump station usage.

Because hourly demand is supplied mainly from the pumping station, an hourly water shortage occurs between the pumping station’s maximum flow rate of 250 m³/h and hourly demand rates of 215 to 445 m³/h. The hourly water balance problem is treated using buffer variables that incur high penalty costs for not meeting an hourly water balance.

In Case III, the pumping station is forced to operate during the high electrical tariff hours, causing the annual operating cost to be higher by approximately 8.4% then in Cases I and II.

Example 2: Complex Example Application

The water distribution system for this application is described in Fig. 6. The system consists of three pressure zones, each having one water tank governing the water pressure in the zone. All pipes are 1 km long and have a Hazen-Williams coefficient of 120. The pipe diameters are given in Fig. 6.

- Pressure Zone a: The zone has two water sources: A pumping station (aP) and a well (aW), both working toward a single water tank (aR). There are four consumers in the zone (aD1–4), junctions aJ2–6, and connecting pipes form two closed supply rings.
- Pressure Zone b: A pumping station (bP) lifts water from tank (aR) to tank (bR), and there is one consumer in the zone (bD).
- Pressure Zone c: A pumping station (cP) lifts water from tank (aR) to tank (cR), and there is one consumer in the zone (cD).

The water distribution system was solved using the water balance minimal cost LP model combined with hydraulic head-loss constraints including a minimal head constraint of +80 m for hours when demand is not zero at demand nodes aD2–4.

Results

The resulting annual operating cost for the system is 789,957 NIS/year. The solution was achieved within 14 min and after five Stage III iteration steps (maxErr = 0%).

The resulting operation of the system over an average week in July is shown in Fig. 7. The three water tanks operate as daily volumes, filled by the pumping stations and well during low electrical tariff periods and emptying to consumption during high tariff periods. Between Sunday and Wednesday, the tanks volumes are fully used. On Thursday, the demand flow rates are the highest of the week and the pumping stations work most of the hours.

![Fig. 5. Basic example application—Pump station and demand node hourly flow rates, average week in June (Case III)](image-url)
directly to demand with only a few hours left in the day to fill the tanks. During Friday and Saturday, all the daily hours have a low electrical tariff charge, meaning that the pumping stations can be constantly operated directly to demand, maintaining maximum water tank volumes. There is an offset between the demand hours and the pumping station’s working hours. This offset is balanced by the water tanks as described previously.

Fig. 8 presents the systems operation on Sunday of an average week in July in Zone a. A water balance offset can be seen between 8 to 11 p.m., the electrical tariff is low and the pumping stations operate to fill the three water tanks. During the hours of 4 to 8 p.m., the electrical tariff charge is moderate with no consumer demand, the pumping stations are off, and the water tank water volume remains constant. During the hours of 2 to 4 p.m., the flow rates to pumping stations bP and cP are further reduced to allow a minimal water head constraint.

During the hours 12 a.m. to 6 a.m. when the electrical tariff is moderate with no consumer demand, the pumping stations are off, and the water tank water volume is constant. During the hours of 8 to 11 p.m., the electrical tariff is low and the pumping stations operate to fill the water tanks.

In Figs. 7 and 8, it is noticeable that the operation of pumping station aP is preferable to the operation of well aW. This is because the water level in the well is significantly lower than the source to the pump (aSp). The nominal energy consumption for the well is much higher and thus is used only for a few hours per day.
Fig. 7. Complex example application—Flow rates and heads at nodes for optimal operation, average week in July

Fig. 8. Complex example application—Flow rates and heads at nodes for optimal operation, July, Sunday, Zone a
Conclusions

The suggested algorithm enables iterative linearization of increasing or decreasing convex nonlinear equations and incorporating them into LP optimization models. The algorithm was successfully demonstrated on the Hazen-Williams increasing convex head-loss equation combined with an LP optimal operation water supply model. The resulting optimal solution combines minimal annual electrical operating costs and hydraulic constraints.

In many water distribution systems, a pure water balance model will suffice for optimal pump operation scheduling for minimal operating costs. However, in some cases, for example an undersized system (pipe diameters are undersized relative to demand flow rates), the head loss along the system may have a significant effect on the actual flow rates a pumping station may deliver or the gravitational flow rates actually possible from the water tanks to the consumers. Also, the head in the system may vary between the hours of the day depending on the consumer demand and the number of working pumps. In an undersized system, a water balance model may result in a nonrelevant operating scheme for the pumps due to undeliverable flow rates. Also, water head constraints like minimum head at consumers or non-negative pressures at junctions may exist in the system, constraints that may only be approximately implemented.

The algorithm can be used on other convex equations of the form $RX^n$ as follows:
1. Separating the nonlinear variable part of the equation $X^n$ from the constant part of the equation $R$.
2. Defining an expected domain along the convex curve in which there is a high probability the solution will be found.
3. Setting the coefficients (i.e., $A$ and $B$) of a linear line equation $(AX + B)$ to best suit the convex curve $X^n$ within the domain.
4. Building and running the optimization model in which the nonlinear convex part of the equation is replaced with the linear line equation in the form of $(AX + B)R$.
5. Setting an anchor point along the convex curve within the domain that linearly best splits the convex curve.
6. Using the resulting optimal solution outcome from the initial model run as starting conditions, and iteratively correcting the $A$ and $B$ coefficients for a line intersecting the previous iteration step to result in the anchor point.
7. The algorithm terminates when an optimal solution is attained and a satisfactorily approximation is received.

The suggested method enables a nonlinear convex problem to be addressed and solved as a LP problem, which results in short solution times. Future implementation in the field of water distribution systems’ optimal operation models include linearization of pump units pumping curves and the linearization of chlorine decay equation over time. Both of these are currently explored in extensions of the current study.

Notation

The following symbols are used in this paper:

- $A_{i,j}$, $B_{i,j}$ = linear equation coefficients constants $(AQ + B)$;
- $dH_{i,j}^p$ = artificial head gain or loss in pipe variable (m);
- $cu$ = annual cost result from current iteration step (NIS);
- $D = group of all demand at nodes$;
- $D_{i,j}$ = pipeline diameter constant (mm);
- $De_{i,j}^d$ = consumer demand at node constant (m$^3$/h);
- $d = demand node index$;
- $dH_{i,j}^p$ = head loss in pipe variable (m);
- $H_{i}^{max}, H_{i}^{min}$ = water tank head boundaries constant (m);
\[ H_i = \text{water head at water tank variable (m)}; \]
\[ H_{Ci,j} = \text{Hazzen-Williams head-loss coefficient constant (-)}; \]
\[ i, j = \text{index of pipe origin and destination nodes}; \]
\[ \text{maxErr} = \text{maximum error between the } Q_i^{1.852} \text{ and } (AQ_r + B) \text{ variables}; \]
\[ N = \text{group of all node indexes}; \]
\[ P = \text{group of all pump nodes}; \]
\[ PE_p = \text{pump nominal energy consumption constant (kWh/m^3)}; \]
\[ PQ_p = \text{pump nominal flow rate constant (m^3/h)}; \]
\[ P_{neg_{i,j}} = \text{negative penalty flow rate change per pipe per hour variable (m^3/h)}; \]
\[ P_{pos_{i,j}} = \text{positive penalty flow rate change per pipe per hour variable (m^3/h)}; \]
\[ (P_{f_{i,j}})_r = \text{previous iteration flow rate constant (m^3/h)}; \]
\[ p = \text{pump station node index}; \]
\[ pr = \text{annual cost result from previous iteration step}; \]
\[ Q_{f_{i}}^{m} = \text{anchor flow rate constant (m^3/h)}; \]
\[ Q_{f_{i}}^{m_{max}} = \text{maximum expected flow rate constant (m^3/h)}; \]
\[ Q_{f_{i}}^{j} = \text{flow rate in pipe variable (m^3/h)}; \]
\[ (Q_{f_{i}}^{j}) = \text{resulting flow rates derived from the previous iteration step}; \]
\[ R = \text{group of all water tank nodes}; \]
\[ R_{i,j} = \text{pipe resistance’s coefficient constant}; \]
\[ r = \text{water tank index}; \]
\[ T = \text{group of all time indexes (1...8,760)}; \]
\[ TH_p = \text{total pump head variable (m)}; \]
\[ Tar_{p} = \text{hourly tariff charge (NIS/kWh)}; \]
\[ t = \text{index of hours}; \]
\[ V_{r_{max}}^{\text{max}}, V_{r_{min}}^{\text{min}} = \text{water tank water volume boundaries constant (m^3)}; \]
\[ V_{i}^{r} = \text{water volume in water tank variable (m^3)}; \]
\[ W_p = \text{pump variable speed change (-)}; \]
\[ \alpha = 0.7 \text{ (-), Method A, } \alpha Q_{f_{i}}^{m_{max}} \text{ constant}; \]
\[ \beta = 0.44 \text{ (-), Method B, } \beta Q_{f_{i}}^{m_{max}} \text{ constant}; \]
\[ \gamma = \text{constant to prevent division by zero (-)} [\text{Eq. (6)}]. \]

References


Lasdon, L. S., and Waren, A. D. (1986). GRG2 user’s guide, Univ. of Texas, Austin, TX.


