A successive linear programming scheme for optimal operation of water distribution networks

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ABSTRACT

Optimal operation of water distribution systems is a well explored problem defined as finding the scheduling of pumping units over time which minimize cost while maintaining flow, pressure, and tank water levels constraints. One of its major complexities is the inherent non-linearity and non-smoothness relationship of the headloss equation (e.g., the Hazen Williams or Darcy Weisbach formulas). This study suggests a method for the linearization of the Hazen-Williams headloss equation which enables the non-linear hydraulic problem to be addressed and solved as a linear programming scheme. The methodology is demonstrated on a small illustrative example.

1. INTRODUCTION

Coping with non-linear optimization models through linearization is a well known explored challenge. This study presents a methodology to generally deal with this difficulty in case the objective function and/or constraints are convex.

An example of such a problem is the optimal operation problem of a water distribution system in which pump schedules are searched for minimizing operational costs while maintaining flow, pressure, and tank water levels. The headloss relationship between flow and head (e.g., the Hazen-Williams or Darcy Weisbach formulas) create non-linear convex equations, thus forming a non-linear and non-smooth (as flow can reverse) optimization problem. If only the headloss equation would hold a linear relationship between flow and head, then the optimal operation problem could have been casted in a linear programming framework and efficiently solved. Since flow in pipes are unknown, linearization is problematic as the linearization domains are unknown. This work describes an overall iterative linear programming scheme for dealing with these difficulties.

2. METHODOLOGY

2.1 General

The model developed in this work is based on a GAMS/CBC (Coin-or branch and cut) linear programming (LP) (https://projects.coin-or.org/Cbc) minimal cost optimal operation water distribution system (WDS) model, developed and actively used in Tahal Consulting Engineers Ltd, Israel. The base model’s objective function is minimal annual operating costs derived from pumping station electrical consumption and electrical tariff. The decision variables are pumping station operating schedule and flow of water between nodes. The model addresses annual operation of the system at an hourly basis. The resulting solution is to maximize the
utilization of water tank/reservoir volumes constrained by water balance closure over the optimization period. In this work the non-linear Hazen-Williams (HW) headloss equation is linearly introduced into the base model.

2.2 Linearization of Hazen-Williams headloss equation

The non-linear Hazen-Williams (HW) headloss equation, shown in Eq. (1), is partitioned into two sub equations. Eq. (2) consists of the constant resistant properties of a given pipe. Eq. (3) holds the linearization of the nonlinear flow $Q_{1.852}$ as a linear equation, subject to the coefficients $A$ and $B$. Eq. (4) is the combined linear headloss equation.

$$dH_{ij}^t = 1.131 \times Q_{ij}^{1.852} \times HC_{ij}^{-1.852} \times D_{ij}^{-4.87} \times 10^9 \times L_{ij} \quad \forall \ i, j \in N , t \in T$$  \hspace{1cm} (1)

$$R_{ij} = 1.131 \times HC_{ij}^{-1.852} \times D_{ij}^{-4.87} \times 10^9 \times L_{ij} \quad \forall \ i, j \in N$$  \hspace{1cm} (2)

$$Q_{ij}^{1.852} = A_{ij}^t \times Q_{ij}^t + B_{ij}^t \quad \forall \ i, j \in N , t \in T$$  \hspace{1cm} (3)

$$dH_{ij}^t = (A_{ij}^t \times Q_{ij}^t + B_{ij}^t) \times R_{ij} \quad \forall \ i, j \in N , t \in T$$  \hspace{1cm} (4)

where: $t =$ time index (hrs), $T =$ time domain (1...8760), $i,j =$ index of pipe origin and destination nodes, $N =$ node domain, $dH_{ij}^t =$ headloss in pipe variable (m), $Q_{ij}^t =$ flow rate in pipe variable (m$^3$/hr), $HC_{ij} =$ Hazen Williams headloss coefficient constant (-), $D_{ij} =$ pipe diameter constant (mm), and $L_{ij} =$ pipe length constant (m). $R_{ij} =$ Resistant coefficient constant. $A_{ij}^t, B_{ij}^t =$ linear equation coefficients constants.

2.3 Determining the $A$, $B$ and $C$ coefficients

The pipe resistance matrix $R$ is calculated using Eq. (2) before the first iteration step as constant. The linear equation $A$ and $B$ coefficients are calculated before each iteration step as constants using one of the following two methods.

**Method A** - creates an initial linearization of the $Q_{1.852}$ curve through the origin and $Q_{ij}^{\text{max}}$ using Eq. (5). $Q_{ij}^{\text{max}}$ is the maximum expected flow rate in a pipe for a maximum expected flow speed ($v = 2$ m/sec), $Q_{ij}^{\text{max}}$ is not an upper bound constraint. The $A_{ij}^t$ and $B_{ij}^t$ coefficients are found for a linear line intersecting two points: the origin [0, 0] and the point $[\alpha Q_{ij}^{\text{max}}, (\alpha Q_{ij}^{\text{max}})^{1.852}]$, as described in Fig. 1 (line A). The value of $\alpha = 0.7$ was optimally found to best represent the above domain linearly while minimizing accumulative error between the linear line and the $Q_{1.852}$ curve.

**Method B** - Iteratively creates a linearization of the $Q_{1.852}$ curve between the points $(Q_{ij})_{r}^{\text{fix}}$ and $(Q_{ij})_{r}^{\text{fix}}$, using Eq. (6) and Eq. (7), as shown in Fig. 1 (line C). $(Q_{ij})_{r}^{\text{fix}}$ is the resulting flow rate matrix from the previous iteration step and $(Q_{ij})^{\text{fix}}$ is a fixed flow rate point per pipe which is determined relative to $(Q_{ij})^{\text{max}}$. $(Q_{ij})^{\text{fix}}$ serves as a constant anchor to which the linearization is made throughout all the iteration steps. The anchor maintains a positive minimal slop value for the linear line when a value of $(Q_{ij})_r$ is close to zero ($(Q_{ij})_r << (Q_{ij})^{\text{fix}}$) and steepens the slop for high $(Q_{ij})_r$ flow rates ($(Q_{ij})_r >> (Q_{ij})^{\text{fix}}$). The $(Q_{ij})^{\text{fix}}$ point was found to be optimally located at $(Q_{ij})^{\text{fix}} = \beta Q_{ij}^{\text{max}}$. The value of $\beta = 0.44$ was optimally found for minimal accumulative error between the $Q_{1.852}$ curve and the two linear lines connecting $(Q_{ij})^{\text{fix}}$ and $(Q_{ij})^{\text{max}}$ and connecting $(Q_{ij})^{\text{max}}$ and the origin, as show in Fig. 1 (line B). If the linearization of the $Q_{1.852}$
curve were to be made directly between $Q_r$ and the origin (not using $Q_{fix}$) then for $Q_r$ values that are close to zero, the slope of the linear line approaches perpendicular with the x-axis ($A_{i,j}^1 \rightarrow 0$), making it near impossible, in the following iteration step, for the model to increase the ($Q_{i,j}^r$) flow rate (to gain headloss between nodes) as the relation between flow rate ($Q_{i,j}^r$) and headloss ($dH_{i,j}$), given in Eq. (4), becomes negligible.

![Figure 1. Linearization of $Q^{1.852}$ for different cases (pipe diameter 250mm)](image)

$$A_{i,j}^1 = \left( \frac{\alpha Q_{i,j}^{max}}{Q_{i,j}^{max}} \right)^{1.852} ; \quad B_{i,j}^1 = 0 \quad \forall \ i, j \in N , t \in T$$

(5)

$$A_{i,j}^1 = \left[ \left( Q_{i,j}^r \right)_t + \gamma \right]^{1.852} - \left( Q_{i,j}^{fix} \right)^{1.852} ; \quad \begin{cases} (Q_{i,j}^r)_t \neq Q_{i,j}^{fix} & \gamma = 0 \\ (Q_{i,j}^r)_t = Q_{i,j}^{fix} & \gamma = 0.01 \end{cases} \quad \forall \ i, j \in N , t \in T$$

(6)

$$B_{i,j}^1 = (Q_{i,j}^r)_t \times \left[ \left( Q_{i,j}^r \right)_t \right]^{0.852} - A_{i,j}^1 \quad \forall \ i, j \in N , t \in T$$

(7)

where: $\alpha = 0.7 (-)$, $Q_{i,j}^{max}$ = maximum expected flow rate constant (m$^3$/hr), $Q_{i,j}^{fix}$ = anchor flow rate constant (m$^3$/hr), $\gamma$ = constant to prevent division by zero.

2.4 Headloss linearization algorithm

The proposed method incorporates the following stages:

**Stage I – LP model**

The purpose of this stage is to serve as filter to whether the problem is at all feasible, see Fig. (2). The model is solved using the water balance LP model with no hydraulic constraints. If the problem is infeasible the algorithm stops and the user is flagged that the problem is infeasible. If an optimal solution is found the algorithm proceeds to stage II.
Stage II – Initial Linearization (using Method A)

The purpose of this stage is to create initial flow rates which will serve as starting conditions for stage III, see Fig. (2). The A, B and R coefficients are calculated according to Method A. If the problem is infeasible the algorithm stops and the user is flagged.

Note that the resulting flow rates from stage II may differ significantly then the flow rates resulting from stage I, due to the hydraulic constraints. Therefore stage II results make preferable starting conditions for Stage III.

Stage III – Main Linearization (using Method B)

The flow rate matrix \( (Q^1_{i,j})_r \) found in stage II serves as the \( Q_r \) matrix for the first iteration step in the current stage, see Fig. (2). The \( A^t_{i,j} \) and \( B^t_{i,j} \) coefficients are calculated according to Method B. The model is solved resulting in a new matrix of flow rates \( (Q^t_{i,j})_r \). Using Eq. (8) The maximum error between the headlosses calculated using Eq. (1) and using Eq. (4) is found for each pipe for each hour. While \( \text{maxErr} > 0 \) the iterative procedure repeats. The process successfully stops when \( \text{maxErr} = 0 \) or fails if a maximum number of iteration steps is reached.

\[
\text{max Err} = \max \left\{ \left| \left( Q^1_{i,j} \right)_r ^{1.852} - \left| A^t_{i,j} \left( Q^1_{i,j} \right)_r ^{1.852} + B^t_{i,j} \right| \right| \times 100 \right\} \quad \forall \ i, j \in N , \ t \in T
\]
Figure 2. Headloss linearization algorithm

3. EXAMPLE APPLICATION

3.1 General

The Methodology is demonstrated on a small illustrative water distribution system, See Fig (3). Results are demonstrated for an average week in June. The system includes an unlimited water source node with a head of +50 m (a.b.l). A pumping station node with a maximum flow rate of 250 m³/hr and TH of 80 m. A tank node with an operational volume of 2,000 m³ and a constant water level of +100 m (a.b.l). A junction node. A demand node with an annual consumption of 1.0 mil-m³/year and a minimal supply head requirement of +90 m. The demand nodes monthly distribution is (Jan to Dec in %): [6.3, 6.3, 7, 7.9, 9.2, 9.9, 10.8, 10.6, 9.7, 8.4, 7.3, 6.6]. For each month the average weekly demand is calculated by dividing monthly demand by the number of weeks in each month. Demand node has a daily demand distribution of (Sun to Sat in %) [15.7, 14.3, 12.9, 13.6, 17.9, 16.4, 9.2] and an hourly demand of 10% between 07:00 to 16:00 (10 hours per day). All pipes are 1.0 km long with a diameter of 250 mm and a Hazen–Williams coefficient of 120. Electrical Tariff is given in NIS/kWhr (NIS = new Israeli shekels). The Tariff is grouped into three annual periods: Summer (Jul-
Aug), Winter (Dec-Feb), Intermediate (Mar-Jun, Sep-Nov). The Tariff is grouped into three daily periods (Sun-Thu, Fri, Sat). According to the session group and the daily period the hours of the day are grouped into three hourly periods: Low rate (marked S), Average rate (marked G) and Peak rate (marked P).

3.2 Case I – Water balance

The above system was solved using the water balance minimal cost LP model with no hydraulic constraints. Resulting minimal annual operation cost is 97,225 NIS/year. The demand hours are mostly during peak tariff hours and the pumping station is mostly operated during low tariff hours (at 260 m3/hr), see Fig (4-I). As a result, the tank is fully filled during low tariff rates and fully emptied during high tariff hours, see Fig (5-I).

3.3 Case II – Water balance combined hydraulic headloss constraints

The above system was solved like Case I but with hydraulic headloss constraints. The resulting minimal annual operation cost is 97,225 NIS/year. The behaviour of the pumping station and tank are the same as in case I, see Fig (4-II), Fig (5-II). Max Error between HW headloss equation and linear HW headloss was 0%. The head at the demand node varies between +109 m and +53 m, see Fig (6-II). High head at demand node occurs when pumping station is working to fill the tank and the low head conditions occur when the pumping station is not working and the supply to the demand node is received from tank.

3.4 Case III – Water balance combined hydraulic headloss and minimal head constraints

The above system was solved like case II including a minimal head constraint at the demand node of +90 m for hours when demand is not zero. The resulting minimal annual operation cost is 105,905 NIS/year. Max Error between HW headloss equation and linear HW headloss was 1.67%. To maintain the minimal head constraint at the demand node, the pumping station was activated during demand hours directly to supply, see Fig (4-III). Because almost all demand is supplied by the pumping station the tank's volume is mostly unused, see Fig (5-II). Because hourly demand is supplied mainly from pumping station, an hourly water shortage occurs between the pumping station's flow rate of 250 m3/hr and hourly demand rates of 212.52 m3/hr to 413.49 m3/hr. The hourly water balance problem is treated using buffer variables with high penalty cost (buffer variables are not discussed in current paper). The pumping station is forced to operate during the high tariff hours causing the annual operating cost to be higher by about 8.9% the operating costs in case I and II (105,905 NIS/year compared to 97,225 NIS/year).

Figure 3. Example Application – Imaginary water distribution system
4. CONCLUSIONS

The suggested algorithm enables iterative linearization of increasing or decreasing convex non-linear equations and incorporating them into LP optimization models. The algorithm was successfully demonstrated on the Hazen-Williams increasing convex headloss equation combined with an LP optimal operation water supply model. The resulting optimal solution combines minimal annual electrical operating costs and hydraulic constraints.

The suggested method enables non-linear convex problem to be addressed and solved as an LP problem which results short solution times. Future implementation in the field of water distribution systems optimal operation models include: linearization of pump units pumping curves and the linearization of chlorine decay equation over time. Both of these are currently explored in extensions of the current study.

5. REFERENCES

"COIN-OR/CBC home page." https://projects.coin-or.org/Cbc.