Optimal stocking in intensive aquaculture under sinusoidal temperature, price and marketing conditions

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1. Introduction

1.1. General

There is considerable literature regarding the management of biomass in aquaculture. General aquaculture management tools, such as AquaFarm (Ernst et al., 2000), can be used to simulate the outcome of different management strategies under a wide range of conditions. They are used to compare different management scenarios but are not designed to deal directly with particular management optimisation problems. Such problems are often solved by means of operational research (OR) tools, the process being described as follows: “OR in aquaculture integrates a biological model of growth of the species, as a function of body weight, water temperature, feed etc., and an economic model, linking the biological production process to the market through input and output prices and resource constraints” (Bjørndal et al., 2004).

The range of optimisation problems in aquaculture is very wide, resulting from the range of species, modes of operation and local conditions. Quite a few studies focus on marine net-cages (Forsberg, 1996; Petridis and Rogdakis, 1996; Forsberg and Guttormsen, 2006; Hernández et al., 2007), but other systems, such as (land-based) recirculating aquaculture systems (RAS; Halachmi et al., 2005; Halachmi, 2007), or fertilized ponds (Yi, 1998) are also considered. Dynamic models based on differential equations (Corey et al., 1983; Lupatsch and Kissil, 1998; Hernández et al., 2003) or Markov processes (Forsberg, 1996) have been used to describe the species population, and queuing theory (Halachmi, 2007) and optimal control (Hernández et al., 2007) have been used to solve the problems, emphasising various restricting conditions (constraints), such as temperature (Cacho et al., 1991; Hernández et al., 2007) and market price (Forsberg and Guttormsen, 2006). Management decisions involve timing of stocking and harvesting (Forsberg, 1996) as well as ration size (Hernández et al., 2007). Most studies consider a single cohort (Petridis and Rogdakis, 1996; Hernández et al., 2007) or a single tank (Hochman et al., 1990), while a few consider a rotation as well (Pascoe et al., 2002). References to earlier studies may be found in Cacho (1997) and in the other cited papers.

The present paper deals with yet another problem. It reflects the need for guidance regarding the stocking strategy of land-based intensive fish farms. A simplified model of such a system is developed, based on assumptions which are mostly compatible with the intended practice of gilthead sea-bream (Sparus aurata) production in recirculating aquaculture systems (RAS). A concise description of the system model is as follows: The fish population is composed of uniform-size daily cohorts (batches), all stocked and harvested at the same initial and final sizes throughout the year, and fed to satiation during their residence in the system. The physical
1.2. Basic model

Growth of fish is often represented by a multiplicative function of the form

$$G = \frac{dW}{dt} = f_1(W) f_2(T) f_3(Z),$$

(1)

where $W$ is live body mass (BM) of a single fish (in g[BM]/fish), $T$ is water temperature (K), $Z$ is feed ration (g[feed]/(fish d)), and $t$ is time (Hernández et al., 2003). In the present study, following common practice in RAS, feeding is to satiation, and hence the third factor may be omitted. Locally established feed consumption (to satiation) and growth functions for gilthead sea-bream (Lupatsch and Kissil, 1998) are

$$F = a_f W^{b_f} \exp\{c_f(T - T_c)\}$$

(2)

and

$$G = \frac{dW}{dt} = a_g W^{b_g} \exp\{c_g(T - T_c)\};$$

(3)

where $F$ is feed intake rate of a single fish (g[feed]/(fish d)), $T_c = 273.2$ is the freezing point temperature (K), and the other symbols are positive constant coefficients (see Nomenclature, below). In general for fish, the size exponent $b_c$ is between 0.5 and 0.7 (Reiss, 1989, p. 41; Jobling, 1994, p. 170), while $b_f$ is somewhat larger, resulting in a diminishing feed utilization efficiency with age. The temperature response is a derivative of the Arrhenius equation, and the value of its coefficient, $c_f$ or $c_g$ in our case, is around 0.07/K (rates doubling every 10 K), varying little among species (Gillooly et al., 2001).

Assuming that the temperature trajectory (sequence) and the initial and final sizes of the fish are given, and noting that feeding is to satiation, the only control available to the grower, according to this model, is the stocking rate (other management tools are fixed). In the simple case where the temperature and the price of fish are invariant with time, the best stocking strategy, intuitively, is to operate the system at steady-state, which means a constant and economic environments have an annual period, represented by sinusoidal trajectories (sequences) of water temperature, price of marketable fish and marketing quota. There is a constraint on the water treatment capacity (oxygen supply and metabolite removal), expressed in terms of feeding rate, which is constant over the year. This constraint is similar to the ‘feed quota’ imposed in Norway for marine cages (Forsberg, 1999), except that there the constraint is annual and here it is daily, which makes the present study significantly distinct. At this stage a small producer is considered, whose rate of supply does not affect the market price.

The model under investigation is not quite realistic: It is implicitly assumed that fish of all sizes are mixed in a single tank (container, pond), neglecting competition and predation. Furthermore, stocking and harvesting are considered to be essentially continuous. In reality fish cohorts must be separated both spatially and temporally. Other assumptions, such as zero mortality and cohorts of uniform size are also introduced to further simplify the analysis. As a result, the predictions of this study indicate the potential yield and profit and may serve only as reference to actual performance, and as guidelines to strategies for more realistic cases. Note, however, that by focusing on the difference between alternative strategies, many of the simplifying assumptions cancel out or may be corrected for. Finally, in Section 4.2 an attempt is made to compare between the idealised (simplified) model and a more realistic situation. In summary, while the detail of the current model is minimal, the broad picture and the qualitative results seem to be credible. Some possible future extensions are indicated in Section 5.
stocking rate. In this scenario, and neglecting fish mortality, each
day of the year $n$ fingerlings are introduced into the system and $n$
marketable fish are removed. Under more realistic conditions,
where the physical and economic environments vary with time,
the optimal stocking rate is expected to also vary. One such non-
steady case is easy to analyse: Suppose that a certain fish can fetch
a price only over a very short period of time (e.g., the week
preceding a certain holiday). The obvious optimal solution is to
stock all the fish over a similarly short period and at such a time as
will make the fish marketable exactly when required. In general,
however, the optimal stocking solution is not obvious.

A first approximation of the annual cycle of a restricting condition
is the first harmonic. It is often valid for water temperature
(Hernández et al., 2007; Seginer et al., 2008) and may fit market
prices (Forsberg and Guttormsen, 2006). When the restrictions are
sinusoidal, it is not unreasonable to assume that the optimal
stocking solution is also close to sinusoidal. The present study
attempts, therefore, to find the best sinusoidal stocking strategies for
a range of plausible restrictions. It is shown that a properly chosen
sinusoidal stocking trajectory often produces considerably better
results than uniform stocking. In the following sections the optimisation problem is first formulated mathematically. Sample cases are then solved, initially for one restriction at a time and then for combinations of restrictions.

2. Optimisation problem

2.1. Sinusoidal restricting conditions

The optimal stocking trajectory (sequence) should maximize the annual net-income (profit) of the grower, subject to one or
more of the following three sinusoidal restricting conditions:

1. Water temperature, $T$

   \[ T(t) = T_m + T_a \sin \left( \frac{2\pi (t - \phi_T)}{365} \right) \]  

   where $T_m$ and $T_a$ are the mean and amplitude of the water
   temperature trajectory ($K$), $t$ is time, and $\phi_T$ is the phase-shift (time
delay; days).

2. Fish price, $P$

   \[ P(t) = P_m + P_a \sin \left( \frac{2\pi (t - \phi_P)}{365} \right) \]  

   where $P$ is the market price of fish ($$/kg[BM]) and the structure of
   Eq. (5) is analogous to that of Eq. (4). Temperature and fish price are
   both exogenous system variables (not controllable by the grower),
   and Eqs. (4) and (5) are restricting conditions on these variables.

3. Marketing goal, $M$

   \[ M(t) = M_m \left[ 1 + \alpha_M \sin \left( \frac{2\pi (t - \phi_M)}{365} \right) \right] \]  

   where $M$ is a marketing goal ($kg[BM]/(m^3[tank] d))$. $M_m$ is the
   mean marketing rate, $\alpha_M$ is a relative amplitude and $\phi_M$ is a phase-
shift (days). Note that the form of this restriction is hypothetical,
introduced to show the potential scope of the proposed approach.

The marketing goal trajectory is obtained (in an idealised situation)
by negotiation between the grower and a distributor (wholesaler,
marketing chain). First they agree about the values of $\alpha_M$ and $\phi_M$.
Based on these values, and using the method described below, the
grower estimates the mean supply rate, first per unit tank volume,
$M_m$, and then for the fish-producing system as a whole. They also
agree about the penalty to be imposed for deviation of the actual
supply rate, $A(t)$, from the marketing goal $M(t)$. At this point a
supply contract is signed between the grower and the distributor.

2.2. Penalty and goal functions

The design of the penalty function, such that proper short term
and long term incentives are provided to grower and distributor, is
beyond the scope of this paper. Here a simple diminishing-returns
function is used, where up to the marketing goal, $M(t)$, there is no
penalty and only surplus production is penalized. A simple
formulation of this kind of penalty is

\[ I(t) = P(t)A(t) \text{ if } A(t) \leq M(t) \]  

and

\[ I(t) = P(t)M(t) \left[ 1 + \frac{1}{p} \left( 1 - \exp \left( - \frac{p}{M(t)} (A(t) - M(t)) \right) \right) \right] \]  

otherwise,  

where $I(t)$ ($$/m^3[tank] d)) is the daily gross income, namely
the payment by the distributor to the grower for the supply of $A(t)$ ($kg[BM]/(m^3[tank] d))$, and $p$ is a penalty factor. When $p = 0$,
there is no penalty (and the payment is proportional to
the amount supplied), while $p \to 0$, the payment for the
excess production (beyond $M(t)$) shrinks to zero. The normalized
form of the function $I(A/M)$, for a range of $p$, is shown in Fig. 1.
It will be shown below that the simplified penalty function
does not affect the overall range of stocking solutions (Section
4.3).

The grower has to deduct from the gross income costs which are
proportional to the rate of production (such as costs of feed,
fingerlings, handling), represented by $C$/kg[BM], as well as the
cost of space (rent, management, water treatment, etc.), repre-
septed by $R$/m^3[tank] d). The annual goal (objective) function,
$J$/m^3[tank] y), to be maximized per unit volume of rearing
volume, is therefore

\[ J = \int_0^{365} \left( I(t) - CA(t) - R \right) dt \]  

2.3. Feeding and fish size constraints

A feeding constraint is imposed on the growing process: At no
time should the feeding rate, $\Phi$, (kg[feed]/m^3[tank] d)), be higher
than a certain critical level, $\Phi_c$, commensurate with the capacity of
the water treatment equipment (supply of oxygen, removal of
carbon dioxide and ammonia). Thus

\[ 0 \leq \Phi \leq \Phi_c \]  

Fig. 1. Normalized daily gross income, $I/PM$, for fish sold under penalty for over-
supply. Actual supply rate is $A$ and marketing goal is $M$. The non-penalized unit-
price of fish is $P$. The value of the penalty factor $p$ is shown on the right margin.

\[ \begin{array}{c|c|c|c}
\text{Normalized income, } I/\text{PM} & 0 & 1 & 2 \\
\hline
\text{Normalized yield, } A/M & 0 & 0.5 & 1 & 1.5 & 2 \\
\end{array} \]
The solution to the problem is a stocking trajectory that maximizes the goal function (Eq. (9)), while satisfying the feeding constraint (Eq. (10), as well as the relevant temperature, price and marketing restrictions (Eqs. (4)–(6)). The solution trajectory is assumed to be sinusoidal, possibly truncated at \( n = 0 \), namely

\[
n(t) = \max \left\{ n_m \left[ 1 + \alpha_n \sin \left( \frac{2\pi t - \phi_n}{365} \right) \right], 0 \right\},
\]

where \( n \) is the number of fingerlings introduced per day (fish/(m\(^3\)tank \( \times \)d)), and the first argument in the curly brackets is analogous to Eq. (6). Setting the values of \( \alpha_n \) and \( \phi_n \) determines also the value of \( n_m \), via the constraint on the feeding rate. To determine \( n_m \), it is at first set to 1 and the resulting feeding trajectory, \( n(t) \), is computed (\( n(t) \) is the integral over all cohorts of the feed-intake \( F \)). The maximum of this trajectory, \( \Phi_x \), is then adjusted to the constraint \( \Phi_{x}\) of Eq. (10), by selecting an appropriate value for \( n_m \),

\[
n_m = \frac{\Phi_{x}}{\Phi_x}.
\]

which is required for the determination of \( M_m \) (Eq. (6)).

The size of the fingerlings and the size of the marketable fish are assumed here to be constant throughout the annual cycle. Hence, for all batches (cohorts),

\[
W_i \leq W(t) \leq W_f,
\]

where the subscript \( i \) indicates the initial fish size (fingerling), and \( f \) denotes the final (marketable) size.

The computation is simplified considerably by assuming that the fish are fed to satiation. Making this (realistic) assumption, the individual daily cohorts grow independent of each other, even if mixed in the same tank. For each fish the future is determined solely by the temperature trajectory (Eq. (4)) and by \( W_f \).

3. Sample computations

3.1. Outline of computations

A search (using any reasonable search procedure) is conducted over the two-dimensional stocking parameter-space \( \alpha_n \) and \( \phi_n \) (Eq. (11)), for the maximum value of the goal function, \( J \) (Eq. (9)). Initially two 365\(^2\) "unit" matrices are prepared, each with columns for days of the year and rows for cohorts of one fish per day. The values in one matrix are the feed (to satiation, Eq. (2)) required for a given cohort (fish) on a certain day, depending on the current body mass and temperature. The values in the other matrix are the body mass attained by fish with the particular temperature history of their cohort (via Eq. (3)). All cohort trajectories (rows) have an annual periodicity (sequentially delayed by one day and modified according to the changing temperature). The body-mass matrix is used to decide when a cohort is ready to be marketed (the values drop from \( W_f \) to zero on marketing day and jump from zero to \( W_f \) on stocking day).

Next, the unit matrices, which are independent of the chosen stocking strategy \( n(t) \), are multiplied by the vector \( n(t) \) (length 365; \( n_m = 1 \); Eq. (11)), adding up over all cohorts to determine the standing stock (fish biomass) and total feed, \( F \), consumed for the stocking strategy (\( \alpha_n \) and \( \phi_n \)) under consideration. Finally, the mean stocking rate, \( n_m \), is adjusted, so that the maximum feed consumption (for all cohorts on the day with the highest feed demand) becomes equal to the maximum allowable level (Eq. (12)). At this stage, \( M_m \), the mean rate of production (supply, marketing), the annual production (yield) \( Y \) (kgBM)/(m\(^3\)tank \( \times \)y); sum over all daily sales) and the goal function \( J \) (Eq. (9)), can be evaluated from the body-mass matrix and the various costs.

A general purpose programming language, Matlab\(^\text{®} \), has been used to create a program for the computations.

3.2. Parameter values

The parameter values are based partially on well established properties of the system (fish parameters, water temperature, some costs) and partially on examples of plausible economic cases. The potential users of this approach should supply parameter values relevant to their own situation. The parameters used in the sample calculations are presented in Table 1.

4. Computation results and discussion

4.1. Reference case

The first case to be considered is the steady-state situation, for constant temperature (24 °C), constant fish price (5 $/kg[BM]) and

<table>
<thead>
<tr>
<th>Class</th>
<th>Source</th>
<th>Symbol</th>
<th>Meaning</th>
<th>Value(s)</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matabolism</td>
<td>Lupatsch et al. (2003)</td>
<td>( \alpha_r )</td>
<td>Feed intake-rate coefficient</td>
<td>0.029</td>
<td>g/[feed]/g[BM] ( \times ) fish(^{1-8} ) d</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( b_r )</td>
<td>Size exponent, feed</td>
<td>0.6</td>
<td>Dimensionless</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( c_r )</td>
<td>Temperature coefficient, feed</td>
<td>0.06</td>
<td>1/K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \alpha_c )</td>
<td>Growth rate coefficient</td>
<td>0.024</td>
<td>g/[BM] (^{1-8} ) / (fish(^{1-8} ) d)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( b_c )</td>
<td>Size exponent, growth</td>
<td>0.51</td>
<td>Dimensionless</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( c_c )</td>
<td>Temperature coefficient, growth</td>
<td>0.06</td>
<td>1/K</td>
</tr>
<tr>
<td>Fish size</td>
<td>Commercial practice, Israel</td>
<td>( W_i )</td>
<td>Initial size</td>
<td>2</td>
<td>g[BM]/fish</td>
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<tr>
<td></td>
<td></td>
<td>( W_f )</td>
<td>Final size</td>
<td>400</td>
<td>g[BM]/fish</td>
</tr>
<tr>
<td>Feeding constraint</td>
<td>Intensive aquaculture</td>
<td>( \Phi_i )</td>
<td>Critical feeding rate</td>
<td>1</td>
<td>kg[feed]/(m(^3)tank ( \times ) d)</td>
</tr>
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<td></td>
<td>Elat, Israel</td>
<td>( T_m )</td>
<td>Mean water temperature</td>
<td>24 + ( \tau )</td>
<td>K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( T_a )</td>
<td>Amplitude of water temperature</td>
<td>0 or 4</td>
<td>K</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \phi_T )</td>
<td>Phase-shift of water temperature</td>
<td>0</td>
<td>d</td>
</tr>
<tr>
<td>Fish price</td>
<td>Plausible mean **</td>
<td>( P_m )</td>
<td>Mean price</td>
<td>5</td>
<td>$/kg[BM]</td>
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<td></td>
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<td>( P_a )</td>
<td>Price amplitude</td>
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<td>$/kg[BM]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \phi_P )</td>
<td>Price phase-shift</td>
<td>90</td>
<td>d</td>
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<tr>
<td>Marketing agreement</td>
<td>Arbitrary</td>
<td>( \alpha_M )</td>
<td>Relative marketing amplitude</td>
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<td>Dimensionless</td>
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<tr>
<td></td>
<td></td>
<td>( \phi_M )</td>
<td>Marketing phase</td>
<td>30</td>
<td>d</td>
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<td>( p )</td>
<td>Penalty factor</td>
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</tr>
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<td>Production costs</td>
<td>Pilot plant, Elat, Israel</td>
<td>( C )</td>
<td>Cost proportional to production</td>
<td>2</td>
<td>$/kg[BM]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R )</td>
<td>Cost proportional to space</td>
<td>1</td>
<td>$/(m(^3)tank ( \times ) d)</td>
</tr>
</tbody>
</table>

\* Arbitrary time origin.

\** Arbitrary amplitude and phase.
no marketing constraint (Fig. 2, frames 1, 2 and 3). This combination is tagged ‘Case 1’ and may be used as a reference for the other cases. The optimal stocking strategy for the reference case is a constant rate of \( n = 1.32 \) fish/(m\(^3\) tank d) (frame 4). The stocking rate is limited by the maximum feeding rate, which is maintained at the upper limit, \( \Phi_t = 1.0 \) kg[feed]/(m\(^3\) tank d) throughout the year (frame 5). The standing stock (of all cohorts) is a constant 65.8 kg [BM]/m\(^3\) [tank] (frame 6), and the production rate is \( A = 0.525 \) (\( =nW_f \)) kg[BM]/(m\(^3\) tank d) (frame 7). The net-income (profit) rate to the grower is \( j = 0.575 \) (=I-CA-R) $/(m\(^3\) tank d) (frame 8). The restricting conditions, best stocking strategy and annual totals are presented in Table 2 (Case 1, columns 1–14).

**Fig. 2.** Results for Case 1 (reference): constant water temperature and fish price; no marketing condition; steady-state process.

<table>
<thead>
<tr>
<th>Case</th>
<th>Temperature</th>
<th>Fish price</th>
<th>Marketing Stocking</th>
<th>Yield</th>
<th>Profit</th>
<th>Case</th>
<th>Stocking</th>
<th>Yield</th>
<th>Profit</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T_m ) (°C)</td>
<td>( T_a ) (K)</td>
<td>( P_m ) ($/kg[BM])</td>
<td>( P_a ) ($/kg[BM])</td>
<td>( \Phi_t ) (d)</td>
<td>( \alpha_m )</td>
<td>( \phi_m )</td>
<td>( \alpha_n )</td>
<td>( \phi_n )</td>
<td>( Y ) (kg/m(^3) y)</td>
</tr>
<tr>
<td>1</td>
<td>24</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.32</td>
<td>0</td>
<td>191.6</td>
<td>209.9</td>
</tr>
<tr>
<td>2</td>
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<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1.29</td>
<td>0.78</td>
<td>304</td>
<td>187.9</td>
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<tr>
<td>3</td>
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<td>5</td>
<td>2.5</td>
<td>90</td>
<td>0</td>
<td>0.91</td>
<td>1.25</td>
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<td>137.1</td>
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<td>4</td>
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<td>0.13</td>
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<td>5</td>
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<td>0.18</td>
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<td>148.4</td>
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<tr>
<td>7</td>
<td>24</td>
<td>4</td>
<td>5</td>
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<td>0.18</td>
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</tr>
<tr>
<td>8</td>
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<td>0</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.09</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Italic and bold values indicate temperature, price and marketing restrictions. Columns 2–9 describe the problem (Case), columns 10–12 are the optimal *sinusoidal* stocking solutions and columns 13 and 14 are the computed production and profit (net-income). Column 16 is the optimal *constant* stocking rate for the same constraints, resulting in the production and profit of columns 17 and 18. Column 19 is the difference between column 14 (sinusoidal stocking) and column 18 (constant stocking).
4.2. Comparison with a realistic steady-state system

Halachmi (2007) analysed a similar problem: The same species and a similar aqueous environment, except under realistic conditions of a finite number of culture tanks and residence times of weeks in each tank (his 2-4-8 scheme). The main differences between the two (virtual, simulated) systems are summarized in Table 3. The growth rates differ somewhat, but it turns out that using the higher growth rate of Halachmi (2007; Eq. (11)) in conjunction with his slightly larger initial body mass (2.5 g[BM]) and somewhat lower mean temperature (23.4 °C), the required time to marketing (at 400 g[BM]/fish), is about 350 days, very similar to the steady-state residence time (352 d) in the current study.

The two systems differ also in terms of the fish-density constraint (Table 3). Halachmi (2007) uses fish body mass to define it, while the current study uses feeding rate. A conversion may be obtained by assuming that in Halachmi's system the maximum fish-density, 80 kg [BM]/m³[tank] is attained on marketing day. At this time the mass of the individual fish is 0.4 kg [BM]/fish, and the rate of feed consumption, calculated by means of Eq. (2), is about 4.0 g[feed]/(fish d), or 0.8 kg[feed]/[m³[tank] d]. A comparison with the current study, where the constraint is \( \Phi_c = 1.0 \text{ kg[feed]}/(\text{m}^3[\text{tank}] \text{ d}) \), requires, therefore, that Halachmi's results be divided by 0.8 (increased).

The yield computed by Halachmi (2007, Table 3) for a growth cycle of 350 days, is 154 metric tons per year, per 14 tanks of volume 92 m³ each, namely 120 kg[BM]/(m³[tank] y), or 0.82 fish/ (m³[tank] d). Dividing this by 0.8 to correct for the different density limitation (previous paragraph), the mean rate of supply to the market becomes 1.03 fish/(m³[tank] d), 78% of the potential production rate estimated by the current study (1.32 fish/(m³[tank] d)). Most of the missing 22% is due to imperfect utilization of the available tank volume, as maximum density is only reached on marketing day. In terms of net-income (profit) per unit rearing volume, and utilizing the C and R values of the current study, the net-

### Table 3

Comparison between the assumptions of Halachmi (2007) and those of the current study, for ‘steady-state’ operation under constant temperature.

<table>
<thead>
<tr>
<th>Source</th>
<th>Species</th>
<th>Growth rate</th>
<th>Initial body mass</th>
<th>Final body mass</th>
<th>Temperature</th>
<th>Density Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gilthead sea-bream</td>
<td>Eq. (2)</td>
<td>2.0</td>
<td>400</td>
<td>24.0</td>
<td>80</td>
</tr>
<tr>
<td>1. Halachmi (2007)</td>
<td>Experimental data</td>
<td>2.5</td>
<td>400</td>
<td>23.4</td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

![Fig. 3](image-url) Results for Case 2: variable water temperature; constant fish price; no marketing condition.
income is only 83.3 $/(m^3[tank] y), compared with the 209.9 $/(m^3[tank] y) of Case 1 (Table 2). The relative loss is, hence, about 60%. This large loss may be due to an unrealistically high cost of rearing space, $1/(m^3[tank] d) = 365 $/(m^3[tank] y), although it has been estimated here on the basis of the actual costs of a pilot aquaculture plant in Eilat. Part of the missing profit may not be recoverable under commercial conditions, but the comparison certainly justifies a closer look at the alternatives for better utilization of the rearing space (water volume).

4.3. Individual restricting conditions

Constant temperature and steady-state operation are not normally possible, unless artificial water heating is employed. The next example (Case 2; Fig. 3) considers a case where the temperature varies with time ($T_m = 24 + T, T_a = 4 K$), while fish price is constant and the market is neutral (no penalty), as before. The range of temperature (frame 1) is typical of conditions in the National Centre for Mariculture in Eilat, Israel, and covers the data domain on which Eqs. (2) and (3) are based. The best sinusoidal stocking rate (frame 4) is slightly sub-optimal, as can be seen in frame 5, where the feeding rate adheres only approximately to $F_c = 1.0$. If the truly optimal stocking sequence (not exactly sinusoidal) could be found, the economic outcome might be the same as in the reference case. Note that the close-to-constant feeding rate is achieved by opposite phases of the temperature (frame 1) and of the standing stock (frame 6), such that the product of feed ingestion (increasing with temperature) and current stock is roughly constant.

The sinusoidal stocking rate (frame 4) has a peak preceding by $(365 - 304)/61$ days the temperature peak and a mean of 1.29 fish/(m^3[tank] d) (Table 2, Case 2, columns 12 and 10), the latter only slightly lower than for the reference case. Table 2 also shows that the annual production, $Y (kg[BM]/(m^3[tank] y)$, of Case 2 is lower by about 2% relative to the reference case, commensurate with the deviation (frame 5) from the maximum possible feeding rate. The loss of profit is about twice as large (~5%), which is explained by Eq. (9), where the net-income is seen not to be proportional to the yield (this also applies to the loss of profit in Section 4.2). The individual points, below and above the curves in frames 7 and 8, are explained as follows: As the temperature varies over the annual cycle, the residence time of the fish changes gradually from one cohort to the next. As a result, there are days when two cohorts are ready to be marketed, while on some other days no cohort reaches its marketing size. In this example there are nine occurrences of each kind over the year. Note that if the residence time would have been exactly 365 days, the outlying points would not occur. Note also that the general methodology would apply as well to considerably shorter and longer residence times.

The next case to be considered, Case 3, assumes a constant temperature but a variable production price. The particular price phase-shift (here 90 days) has no significance in this case, as $T$ and

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Fig. 4. Results for Case 3: constant water temperature; variable fish price; no marketing condition.
are constant, but it may have an effect when a combination of several restrictive conditions is considered. Fig. 4 and Table 2 show the solution for Case 3. The stocking peak is delayed by \((103 / 90) 13\) days relative to the peak in price (frames 4 and 2), in an attempt to match the production peak (frame 7) with the price peak. This delay agrees with the mean residence time of the fish in the system, which is \((365 / 13) 352\) days, 2 weeks shorter than a year.

There is a period of 75 days during which the stocking rate is zero (Fig. 4, frame 4; Eq. (11)) and a similar period (2 weeks ‘earlier’ or 50 weeks later) when production (marketing) is zero (frame 7). Mean stocking rate, and hence annual production (Table 2, columns 10 and 13), are about 30% lower than for the reference case (steady-state). Nonetheless, by taking advantage of the peak in price, the annual profit (column 14), is actually 16% higher than for the reference case.

The next case (Case 4) explores the effect of a variable-marketing-rate on the stocking strategy of the grower. Fig. 5 shows the effect of the penalty factor \(p\) on the stocking strategy. As long as \(p\) is less than about 5.5, the stocking rate remains constant, namely with zero amplitude \((a_n = 0)\) as in the reference case, while the net-income drops from 210 to below 190 $/(m^3[tank] y) as \(p\) increases. At the critical value of \(p\) (\(\sim 5.5\)), the solution switches (jumps) to a phase-shift of \(\phi_n = 43\) days (not shown in Fig. 5) and the stocking amplitude starts to increase from zero at the critical point, to \(a_n \approx \alpha_M = 0.2\) as \(p\) approaches infinity. Note that the stocking phase-shift of \(\phi_n = 30 + 13 = 43\) days is the result of a 13 days delay beyond the marketing phase (already explained in conjunction with Case 3). For the arbitrary choice of \(p = 16\) (Case 4), the optimal amplitude (at \(\phi_n = 43\) days) is \(a_n = 0.13\) (Fig. 5; Table 2, column 11).

Fig. 5 shows that there are essentially two solution types: (1) A constant stocking rate for the low range of the penalty factor (below the critical \(p\)), where the high productivity (per tank volume) compensates for the penalty resulting from the supply–demand mismatch. (2) A stocking rate with phase (time delay) adjusted to the marketing trajectory (agreement), in order to reduce the penalty when the penalty factor is high. In particular,

Fig. 6. Results for Case 4: constant water temperature; constant fish price; variable marketing condition.
when the penalty approaches infinity (no reward at all for over-production; Fig. 1), actual production, A[t], approaches the marketing goal (agreement), M[t]. Fig. 6, for p = 16, shows an intermediate situation, where the stocking phase corresponds to the marketing phase (frames 7 and 3), but the amplitude is smaller than required for a perfect marketing match (0.13 < 0.20). The trade-off is between maximum utilization of the physical system (frame 5) and adherence to the requirements of the market (frame 7).

It is interesting to note that the particular shape of the penalty function (here Eqs. (7) and (8)), while affecting the profit, is not critical relative to the stocking strategy: Whatever the formulation, a very low penalty factor would result in a constant stocking rate and a net-income close to 210 $(m^3/tank y)$ (left hand side of Fig. 5), while a very high penalty factor would result in production rate very close to the agreement and a net-income close to 173 $(m^3/tank y)$ (right hand side of Fig. 5). Intermediate values of the penalty factor may be converted from one penalty formulation to the other.

4.4. Combinations of restricting conditions

Up to this point only single restricting conditions have been considered. Results for pairs of the individual conditions and for a combination of all three are summarized in Table 2, Cases 5–8. It turns out that each combination has its peculiar properties. Sometimes, adding a restriction reduces the profit below any of the former results (Cases 5 and 6), and sometimes an intermediate profit is obtained (Cases 7 and 8). The main factor behind this range of possibilities is the phase difference (time delay) between the optimal stocking solutions for the individual restrictions. As an example, the phase (Table 2, column 12) of the optimal solution to Case 2 (variable temperature) is 304 days (profit 198.6 $(m^3/tank y)$). The phase of the solution to Case 3 (variable price) is 103 days (profit 242.9 $(m^3/tank y)$). The phase difference is 201 days, about half a year, meaning negative correlation between the two stocking sequences. Therefore, when both restrictions are imposed simultaneously (Case 5), the loss of performance is considerable, the profit dropping to 131.5 $(m^3/tank y)$. How-ever, if the price condition was to be shifted by 304 – 103) 201 days (new phase (90 + 201) 291 days), so that the two individual solutions are exactly in phase (both at 304 days), the combined solution (not shown in the table) is much better than before, producing a profit of 392.1 $(m^3/tank y)$. Unfortunately, the phases of the constraints are given by the prevailing supply–demand situation and cannot be manipulated.

If the solutions to the individual conditions are not in phase, the phase of the combined solution is at some intermediate position, depending on the dominance of the individual conditions. If a penalty, such as p, is involved, changing it moves the solution phase. For example, by increasing the penalty p from zero to infinity, the stocking solution (columns 10–12) of Case 7 moves from that of Case 3 to that of Case 4.

4.5. Constant stocking rate

Columns 15–18 of Table 2 show the results for the best constant stocking rate. Note that there are only two stocking solutions: One for constant temperature ($n_m = 1.32$ fish$(m^3/tank d)$) and another for the variable temperature ($n_{av} = 1.02$). There is a wide variation, however, among the economic outcomes, J, of these cases, which are always inferior to the corresponding best sinusoidal solutions.

Cases 12, 13 and 15 (temperature and price effects) may be interpreted (for the set of inputs used here) as situations where the market requires a constant supply of fish, irrespective of the difficulties to the grower. This is explained as follows: As the residence time is approximately one year long (365 – 13 = 352 days on average), all cohorts are exposed to nearly the same temperature distribution and grow at about the same average rate (but note the few individual points of Fig. 3). This means that a constant stocking rate results in an approximately constant supply of fish to the market.

In general, the results of Cases 12 –18 show that ignoring the effects of the variable conditions, namely stocking at a constant rate irrespective of conditions, may be rather costly (Table 2, column 19).

5. Conclusions

1. Sub-optimal solutions to situations where restricting temperature and market conditions are sinusoidal, can be obtained by assuming that the stocking sequence is also sinusoidal. These solutions, obtained by a simple two-dimensional search, seem credible and are certainly better than the best constant stocking solutions. In particular, the solution for variable temperature manages to maintain an almost perfect utilization of the available water volume. Furthermore, truncated sinusoidal can also produce useful results.

2. Comparison of the idealised steady-state solution with a solution for a more realistic system, where rearing volume is not used to capacity, indicates that production in commercial systems may be about 20% lower than predicted here. This may result in a significant reduction of profit per rearing space (water volume) and deserves a closer look into improved space utilization.

3. The exact formulation of the penalty function does not seem to be critical from the stocking strategy point of view. For all conceivable formulations, if the penalty factor is low, the other restricting conditions (if any) dictate the solution. If, on the other hand, the penalty factor is very high, the production rate must adhere as closely as possible to the marketing agreement to avoid a heavy penalty.

4. The phases of the various restricting conditions affect the success of treating several of them simultaneously. If the individual phases are close, a profitable solution is attainable. However, the three restrictions considered here (temperature, price and market), tend, in practice, to counter each other: Whenever supply is easy due to a favourable temperature sequence, the price tends to drop, or else a marketing restriction must be imposed in an attempt to maintain the price.

5. While the examples considered here are somewhat artificial, they serve to indicate the importance of proper temporal stocking variation. In particular, a constant stocking rate, when inappropriate, may result in a considerable loss of profit.

With view towards future methodological improvements, the sinusoidal solution could possibly be further extended by considering restrictions and solutions described by a small number of harmonics (not just one). Further improvement could be gained by optimisation techniques that are capable of handling any stocking sequence (365 separate stocking decisions). Consideration of additional economic factors, such as the effect of supply on the market price or alternative marketing routes (e.g., frozen fish), may result in more realistic problem descriptions. Refinements such as calculations in terms of present value, inclusion of mortality rate and other considerations may also enhance the results.

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References