Temporal and Spatial Growth of Wind Waves

Vol. 37, Iss. 1; pg. 106, 9 pgs

Abstract (Summary)

A solution of Rayleigh's instability equation, which circumvents the apparent critical-layer singularity, is provided. The temporal and spatial growth rates of water waves exposed to a logarithmic wind profile are calculated and discussed. The findings are similar to previously published results, except for shear velocity-to-wave celerity ratios larger than 2, where the newly calculated growth rates start to decrease after having reached a distinct maximum. The ratio of the spatial to temporal growth rates is examined. It is shown to deviate by up to 20% from the leading-order value of 2. The implications of the growth rate to the modal distributions of energy input from wind to waves, for young and mature seas, and in temporal/spatial growth scenarios, are analyzed.

1. Introduction

The study of the growth of ocean surface waves by the wind blowing over them is often treated as a linear stability problem. Miles (1957) was the first to use Rayleigh's (1880) equation for this problem. When studying linear stability problems, it seems helpful to consider two typical scenarios: one of temporal growth and the other of spatial growth. The shear velocity $U^*$ and the amplitude of the wave $a$ for both scenarios are given by:

(i) the temporal growth scenario

$$\beta = \frac{\alpha}{2}$$

and

(ii) the spatial growth scenario

$$\alpha = \frac{\beta}{2}$$

In (1.1b) and (1.2b), $\beta$ and $\alpha$ are the temporal and spatial growth rates, respectively, and to lowest order, $\beta = \frac{\alpha}{2}$ (Gaster 1962). For the sake of completeness, it is worthwhile to mention that other scenarios of combined growth are, of course, possible. Generally speaking, $\alpha$ and $\beta$ depend on the shear velocity and on the wavenumber $k_0$ [or wave frequency $\omega_0 = (gk_0)^{1/2}$]. However, the actual rate of energy transfer from the wind to the waves depends also on $a_0$ itself. In linear stability studies, one is interested, among others, to detect the fastest growing mode (wavenumber). Here it is suggested to widen this interest and to look also at the fastest energy-accumulating modes under several scenarios. The aim of this paper is twofold:

(i) to investigate the accuracy of the relation $\beta = \frac{\alpha}{2}$ and

(ii) to find which wave modes (i.e., which parts of the spectrum) are absorbing most of the energy from the wind, in both of the aforementioned scenarios and for young/mature sea states.

In this paper we restrict the discussion to gravity waves and assume a steady shear velocity in the air. The only effect of turbulence that enters our derivation is its effect on the mean wind profile.

The mathematical problem is formulated in section 2, and three different methods of solution are discussed in section 3. The issues of comparison between temporal and spatial growth and of the fastest-growing modes (for young and mature seas) are treated in sections 4 and 5, respectively. Conclusions are drawn in section 6.
2. Mathematical formulation

Assuming two-dimensional inviscid and incompressible flows in the water (denoted by subscript \( w \)), as well as in the air (denoted by subscript \( a \)) above it, and relatively small wavy components superposed on steady, leading order given, shear flows, the velocities, pressures, and densities are denoted by

* horizontal air velocity component: \( U^a(z) + u^a(x, z, t) \);
* vertical air velocity component: \( u^a(x, z, t) \);
* air pressure: \( P^0 - g \rho_a z + p^a(x, z, t) \), where \( \rho_a \) is the density of the air;
* horizontal water velocity component: \( U^w(z) + u^w(x, z, t) \);
* vertical water velocity component: \( u^w(x, z, t) \);
* water pressure: \( P^0 - g \rho_w z + p^w(x, z, t) \), where \( \rho_w \) is the density of water, and \( P^0 \) is a constant reference pressure.

\( U^a \) and \( U^w \) are the prescribed unperturbed wind and current, respectively.

Note that lowercase letters indicate fluctuating quantities.

The wavy interface between the water and the air is \( z = \eta(x, t) \). The continuity equations and the linearized equations of motion are

... (2.1a)
... (2.1b)
... (2.1c)
... (2.2a)
... (2.2b)

and

... (2.2c)

In the above equations, the prime denotes differentiation with respect to the vertical coordinate, \( z \).

The systems (2.1) and (2.2) have wavy solutions with wavenumber \( k \) and frequency \( \omega \):

... (2.3a,b)
... (2.3c)
... (2.4a,b)
... (2.4c)

where the auxiliary functions \( f^w \) and \( f^a \) satisfy Rayleigh’s equation.
Note that the real part of a complex quantity represents the physical variable.

For the interface with initial amplitude $a^0$, the linearized kinematic and dynamic free-surface boundary conditions are

\begin{align}
& (2.8a,b) \\
& (2.9)
\end{align}

where $r$ is the surface tension divided by the density of the water.

In terms of the auxiliary functions, (2.8) and (2.9) reduce to

\begin{align}
& (2.10a,b) \\
& (2.11)
\end{align}

where $\rho = \rho_a / \rho_w$.

Equation (2.11) is a dispersion equation, giving the relation between the frequency $\omega$ and wavenumber $k$.

Restricting the discussion to shear flows for which ... tend to zero, the two additional boundary conditions for the auxiliary functions are

\begin{align}
& (2.12a) \\
& (2.12b)
\end{align}

For cases with a negligible current in the water, the solution of (2.5), (2.10a), and (2.12a) is

\begin{align}
& (2.13)
\end{align}

Substituting (2.13) into (2.11) and neglecting the influence of surface tension gives

\begin{align}
& (2.14)
\end{align}

The numerical examples in this paper are restricted to logarithmic wind profiles

\begin{align}
& (2.15)
\end{align}

where $k = 0.41$ is von Kármán's constant, $u^*$ is the so-called shear velocity, and the roughness $z^0$ is given by Charnock's relation with constant $\alpha_{ch}$.
The methods of solution are outlined in the following section.

3. Methods of solution

To find $f^a$, for given $k$ and $\omega$, one has to solve

$$(3.1)$$

with the two boundary conditions

$$(3.2a,b)$$

see (2.6), (2.1Ob), (2.15), and (2.12b).

However, one cannot choose $k$ and $\omega$ freely since they have to fulfill the dispersion relation

$$(3.3)$$

see (2.14) and (2.15).

Owing to the influence of the wind, the frequency $\omega$ and the wavenumber $k$ can have small but important deviations from their values $\omega^0$ and $k^0$ in windless conditions. This fact is made explicit by the notation

$$(3.4a)$$

$$(3.4b)$$

where $\omega^1$ and $k^1$ can be complex and

$$$(3.5)$$

is the windless dispersion relation.

At least three methods of solution are possible; they will be referred to as the singular approach, exact approach, and higher-order solution for spatiotemporal growth—for reasons to become obvious later.

a. The singular approach

In this approach ($\omega$, $k$) are replaced by $\omega^0$ and $k^0$ in (3.1), (3.2a,b) as well as on the rhs of (3.3) and by (3.4a,b) on the lhs of (3.3):

$$(3.6)$$

$$(3.7a,b)$$

$$(3.8)$$

Neglecting the small first term on the lhs of (3.8) and then dividing by $2\omega^0$ gives

$$(3.9)$$

where $c^g = g/2\omega^0$ is the group velocity.
Equation (3.6) is singular at the so-called critical layer, where $U^a = c^a = \omega^a = \omega_0^a = k^a$, is the phase velocity. This singular equation has been solved by Conte and Miles (1959), Janssen (1991), and Beji and Nadaoka (2004), among others.

Taking the imaginary part of (3.9) and recognizing that

$$\omega^1 \text{ or } k^1 \text{ (or both)}$$

are the temporal and spatial growth rates of the amplitude $a$ (i.e., one-half of the wave height, $a > 0$), respectively, one obtains

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.10)$$

Equation (3.10) is rewritten in terms of energy as

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.11)$$

Note that the first/second term on the lhs of (3.11) vanishes for pure spatial/temporal growth conditions. In any case, it turns out that $\omega^1$ or $k^1$ (or both) have imaginary parts so that the original Rayleigh equation (3.1) is actually regular, which leads us to the following two additional methods of solution.

b. The exact approach

Substituting (3.4a,b) into (3.1), (3.2a,b) and (3.3) yields

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.12)$$

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.13a,b)$$

and

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.14)$$

In the above system, $U^a$, $k^a$, $\omega^a$, $\omega_0^a$ are given and $f^a$, $k^1$, $\omega^1$ are unknowns to be found simultaneously. It is quite clear that the system (3.12)-(3.14) has too many unknowns, and either $k^1$ or $\omega^1$ has to be omitted, that is, set to zero. Note that the case $k^1 = 0$ corresponds to pure temporal growth, whereas $\omega^1 = 0$ corresponds to pure spatial growth. It is self-evident that for these two special cases the above regular equation is the exact equation for the problem at hand, whereas the singular approach should be treated as its approximation.

1) TEMPORAL GROWTH

Substituting $k^1 = 0$ in (3.12), (3.13a,b), and (3.14) gives

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.15)$$

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.16a,b)$$

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.17)$$

2) SPATIAL GROWTH

Substituting $\omega^1 = 0$ in (3.12), (3.13a,b), and (3.14),

$$\omega^1 = \frac{\omega_0^1}{k_0^1} \quad (3.18)$$
The method of solution of the exact approach is the same for the spatial and temporal cases and is outlined for the latter. Equation (3.15) is of second order but has to obey three boundary conditions: (3.16a,b) and (3.17). The first two are used while solving the Rayleigh equation (3.15) for a given value of $\omega^{1\wedge}$, whereas the dynamic boundary condition (3.17) is used to obtain the following iteration for $\omega^{1\wedge}$, until a specified number of significant digits remains unchanged. The iteration is started with a "first guess" imaginary value for $\omega^{1\wedge}$.

In solving the ODE (3.15), a large value of $z$ is chosen, typically $k^{0\wedge} Z^{\infty\wedge} = 100$, where (3.16b) is replaced by

Equation (3.15) is solved by stepping from $Z^{\infty\wedge}$ to $z = 0$, using Mathematica's solver, and then normalizing to satisfy the kinematic boundary condition (3.16a). Table 1 demonstrates the convergence of the process, for the example with $\alpha^{\wedge \text{Ch}} = 0.0178$, $u^* = 0.08 c^{0\wedge}$, and $\rho = 10^{\text{sup} - 3\wedge}$.

Note that the deviation of the new (exact approach) results from singular (critical layer) calculations, for the temporal scenario, is less than 10%; see appendix. A similar iterative method was used by Morland et al. (1991) to study another instability problem. The limitation of the exact solution is its inability to treat the combined growth problem, a difficulty which is overcome by the third method of solution.

c. Higher-order solution for spatiotemporal growth

Taking the expansion of $f_a$ to first order in $\omega^{1\wedge}$ and $k^{1\wedge}$:

Equations (3.12)-(3.14) can now be expanded in orders of $\omega^{1\wedge}$ and $k^{1\wedge}$. The leading order gives (3.6)-(3.8). At the next order we have a system for $f_\omega$:

and

and a system for $f_k$:

Expanding (3.14) to $O(\omega^{1\wedge}, k^{1\wedge})$ leads to

where
The imaginary part of (3.28a) reduces to (3.9) at leading order and leads to an evolution equation corresponding to (3.11). The values of $f'^0(0)$, $f'^\omega(0)$, and $f'^k(0)$ can be found without solving the singular Eqs. (3.6), (3.23), and (3.26). Instead, we solve (3.12) and (3.13) with three sets of values for $(\omega_1^1, k_1^1)$ (choosing complex values, to avoid singularity). If the values for $(\omega^1, k^1)$ are small enough, we can extract the above three values of $f'^0(0)$, $f'^\omega(0)$, and $f'^k(0)$ from the three values of $f'^a(0)$ and Eq. (3.22). The computational results for temporal/spatial growth scenarios agree with the results of the exact method.

4. Comparison between spatial and temporal growth conditions

For temporal or spatial growth, respectively: $k$ or $\omega$ is kept constant that is, $(k = k^0$ or $\omega = \omega^0)$, Im{$\omega$} or Im{-$k$} provide the growth rates, and Re{$\omega$} (or Re{&}) are slightly different from $\omega^0$ (or $k^0$). These slight changes as functions of $u^*/c_0$ hardly exceed 6%, as one can see from Fig. 1.

The ratio of the dimensionless spatial growth rate $\alpha$ to its temporal counterpart $\beta$ is given in Fig. 2:

\[ \frac{\alpha}{\beta} \]

Note that the ratio $\alpha/\beta$ varies in the range 1.75 to 2.4, which is within 20% of the value 2 (the ratio of the phase velocity to the group velocity), predicted by (3.9) and by Caster (1962). Note that (3.9) also predicts

\[ \frac{\alpha}{\beta} \]

as is clearly reflected in Fig. 1.

5. On the fastest energy-accumulating modes

Figure 3 gives the newly calculated temporal growth rate for $\alpha^* Ch^* = 0.0144$ and $\rho = 1/800$, together with previously published results from Komen et al. (1994). The main difference between the new result and that of Komen et al. is in the maximum found near $u^* c_0^* = 2$, which identifies a fastest-growing mode (i.e., largest $\beta$) for a given shear velocity $u^*$. Note that the experimental data for large $u^* c_0^*$ is already within the gravity-capillary range, whereas both theoretical lines are for pure gravity waves.

The related, fastest energy-accumulating modes, for four different physical scenarios are addressed next. These scenarios are as follows: (I) temporal growth where all modes have the same initial amplitude $a^* = 0$; (II) temporal growth where all modes have the same initial steepness $c^* = ak$; (III) and (IV) similar to (I) and (II), respectively, but for spatial growth conditions. To achieve the above goal one has to nondimensionalize the dependent variables $d(a^* c_0^*)/dt$ and $d(a^* c_0^*)/dx$ by using $(a^* c_0^*)$, $g$, and $u^*$ for cases I and III and by using $(c^* c/> c_0^*)$, $g$, and $u^*$ for cases II and IV. From dimensional considerations one can show that

\[ (5.1a,b) \]

and

\[ (5.1c,d) \]

Note that different length scales have been used to normalize $a$ and $x$ in (5.1c,d), which is appropriate because of the linear nature of the problem.
The results are depicted in Fig. 4 as functions of the independent variable \(c/u^\ast\); the maxima and the ranges of \(c/u^\ast\) for which the energy-accumulating range is larger than half of the appropriate maximum are given in Table 2.

For young seas, where all modes are assumed to have the same initial amplitude (see cases I and III), most of the energy goes into the very short waves, \(c/u^\ast \in (0.1, 0.7)\) and no substantial difference between the temporal and spatial scenarios is detected. For mature seas, where all modes are assumed to have the same initial steepness \(\epsilon^\ast \in (0^\ast, 0^\ast)\), a profound difference between the two scenarios occurs. For the temporal cases, most of the energy goes into rather long waves \(c/u^\ast > 7\); whereas, for its spatial parallel, the energy goes into a wide range of modes \(c/u^\ast \in (0.5, 17)\).

6. Summary

The study of the growth of waves under the influence of wind using the Miles theory, which was started almost 50 years ago (Miles 1957), is continuing to stimulate the interest of the scientific community. Such studies find their main application in improving the accuracy of wave-forecasting models.

The present paper addressed a few issues, some of which help to obtain new answers, and others open further questions. These issues are mentioned below in order of appearance:

(i) A method of solution, called the exact approach, that circumvents the critical-layer singularity was adopted. The deviations between this method and the standard singular approach were found to be less than 10%.

(ii) The spatial growth rate was computed directly. The ratio of the spatial to temporal growth rates was shown to deviate by up to 20% from the leading-order value of \(c/c^\ast\).

(iii) An efficient higher-order solution method was introduced that closely reproduced the exact approach results.

(iv) The wave modes that extract most of the energy from the wind were found, as one would expect, to depend on the actual sea condition. The profound difference in the range of substantial energy input, between temporal growth and spatial growth for a "mature" sea, seems less intuitive.

Note that the maximum growth rate in Fig. 3 and the maxima in Figs. 4a, 4c, and 4d are for a small \(c/u^\ast\) value. Unless \(u^\ast\) and also \(U^\ast = U^\ast \exp z = 10m^\ast\), are very large (\(U^\ast > 20 m s^{-1}\)), these waves are within the gravity-capillary range. A more detailed study of these maxima for less strong winds will require including the surface tension term.

Last, the auxiliary function \(f^\ast a^\ast\) is illustrated in Fig. 5, which demonstrates how different it is from \(\exp(-k^\ast z)\). The large derivatives for \(z \leq z^\ast\) indicate the possible importance of viscous terms. Nevertheless, for large enough \(z\), the relative deviation of \(|f^\ast a^\ast / e^{-k^\ast z}| = 10(2\pi/k^\ast z)\), is less than 1% for \(z > 10(2\pi/k^\ast z)\).

Acknowledgments. This research was supported by the Israel Science Foundation (Grant 695/04) and by the Fund for Promotion of Research at the Technion.

REFERENCES


Comparison between the Exact and Singular Approaches

The exact method of solution is essentially different from the singular critical layer approach and is expected to provide improved results. In Table A1, new results by the exact approach for the normalized temporal growth rate are compared with those of Conte and Miles (1959), and Beji and Nadaoka (2004), using their normalization, their coefficients $\alpha_{Ch} = 0.0178$, $\rho = 10^{-3}$, and their values of $c_0/U_1$, where $U_1 = u_*/\kappa$.

The deviation of the new (exact approach) results from previous (critical layer) calculations is less than 7%. Conte and Miles, as well as Beji and Nadaoka, do not provide results for values of $u_*/c_0$ larger than 0.4. In this range of larger values of $u_*/c_0$, the method used by Janssen (1991) gives accurate results.

Rerunning the critical layer subroutine, written by Janssen (1991), for $\rho = 0.001225$ and $\alpha_{Ch} = 0.0144$, we have obtained the comparison presented in Table A2.

The new results are within 10% of those obtained by the critical layer method.