Using Lyapunov’s direct method for wave suppression in reactive systems

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Abstract

This paper proposes a new approach for stabilizing a homogeneous solution in reaction–convection–diffusion system with oscillatory kinetics, in which moving or stationary patterns emerge in the absence of control. Specifically, we aim to suppress patterns by using a spatially weighted finite-dimensional feedback control that assures stability of the solution according to Lyapunov’s direct method. A practical design procedure, based on spectral representation of the system and the dissipative nature of parabolic PDEs, is presented.

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1. Introduction and statement

Applications of Lyapunov’s direct method for control of lumped nonlinear systems have been extensively discussed (e.g. [13]). Recent attention in the control literature is focusing on applications of Lyapunov-based methods for distributed systems (see, for example, Refs. [2,4–10]). In this paper we consider a finite-dimensional feedback control aimed to stabilize the homogeneous steady-state solution of a reaction–convection–diffusion (RCD) system. Such systems, typically described by nonlinear parabolic partial differential equation (PDE), may admit a large number of solutions. The control should assure a wide domain of attraction of the desired solution and should suppress other coexisting solutions. The most notable contribution of this paper is in finding the form of linear finite-dimensional control that stabilizes RCD systems by Lyapunov’s direct method. The control parameters, like the number of actuator or sensors and the gain coefficient, affect the size of the domain of attraction.

In this work we employ the following Lyapunov function:

\[ J(t) = \frac{1}{2} \int_0^L x^2(z, t) \, dz > 0 \]  

(1)

to establish stability of the homogeneous solution \( x(z, t) = 0 \) of the closed-loop RCD system \((0 < z < L)\)

\[ \begin{align*}
Le \, x_t + V \, x_z - x_{zz} &= f^*(x, u) + \lambda, \\
x_z(0, t) &= V \left[ x(0, t) - x_{\text{in}} \right], \quad x_z(L, t) = 0, \\
u_t + Vu_z - Du_{zz} &= g(x, u), \\
Du_z(0, t) &= V \left[ u(0, t) - u_{\text{in}} \right], \quad u_z(L, t) = 0,
\end{align*} \]

(2)

(3)

with spatiotemporal control \( \lambda = \lambda(z, t) \). In our interpretation model (2), (3) accounts for an activator \((x = x(z, t))\), which undergoes reaction (expressed as \( f^*(x, u) \)), advection and diffusion, and for a fast inhibitor \((u = u(z, t))\), which may be advected by the flow. We assume throughout the paper that \( x(z, t), u(z, t) \) to be sufficiently smooth on \([0, L]\) and belong to the space of square integrable functions that satisfy the boundary conditions of Eqs. (2) and (3), \( z \) is the spatial variable; \( Le \) is the Lewis number and \( V, D \) are convective velocity and diffusion coefficient, respectively. The kinetics terms are of the form \( f^*(x, u) = f(x) + u, \) \( g(x, u) = -du - \beta x \) where \( d, \beta \) are positive constants and \( f(x) \) is a polynomial in \( x \) of odd degree. We propose that the nonlinear function \( f(x), x \in (-\infty, \infty) \) satisfies the sector conditions \([13]\)

\[ xf(x) \leq Cx^2, \quad f(0) = 0. \]

(4)

This property applies for a polynomial of odd degree in \( x \) with a strictly negative highest order coefficient and \( f(0) = 0 \) (e.g., a cubic, \( f(x) = -x^3 + x \), or a quintic, \( f(x) = -x(x^2 - 1)(x^2 - u^2) \) \([14]\)).
For illustration purpose we use the ideal infinite-dimensional linear feedback control
\[ \lambda(z, t) = -kx(z, t). \] (5)

For practice design we use the finite-dimensional discrete-version realization
\[ \lambda(z, t) = -k \sum_{i=1}^{m} \eta_i(t) \phi_i(z), \] (6)

where \( k > 0 \) is a gain coefficient, \( \eta_i(t) \) are time-dependent functions (manipulated inputs), \( m \) is the number of manipulated inputs and \( \phi_i(z) \) define the spatial distribution of the control. Typically \( \phi_i(z) \) and \( (\phi_i^*)^* \) are eigenfunctions (adjoint eigenfunctions) of the linear operator of (2). We use \( \eta_i = \langle x(z, t), \phi_i^*(z) \rangle \) where \( \langle \cdot, \cdot \rangle \) denotes the integral of the product of above functions with respect to \( z \).

Now, if we can find a sufficiently large gain \( k \) such that the Lyapunov function \( J(t) \) for the closed-loop system can be shown to decay exponentially \( (dJ/dt = xJ, x < 0) \) for all (or some) initial conditions, then controller (5) (or (6)) is said to stabilize system (2), (3) by Lyapunov’s direct method.

This model (Eqs. (2) and (3)) and similar ones have been employed for many years in simulating patterns in chemical systems, especially in reaction–diffusion systems (i.e., \( V = 0 \)). Stationary spatially periodic or even spatially complex solutions may emerge when the activator capacity is sufficiently large (\( Le \gg 1 \)). The inhibitor diffusivity (\( D \)) is not crucial for the establishment of these pattern and it is set to zero [11].

Several recent applications in reaction–diffusion and fluid-dynamics systems derived the methodology for control of a desired spatiotemporal pattern ([16,17] and references therein). The proposed ideal control strategy (5) presents a linear state feedback that shifts the linear upper bound on the system nonlinearity and thus assures that the derivative of the Lyapunov function is negative. The actual physical implementation of such infinite-dimensional control is not practical. Thus, a finite number of sensors and actuators is required to approximate this control. Although distributed systems are described by an infinite-dimensional set of ODEs, the instability of spatiotemporal patterns is typically characterized by a small number of temporal unstable modes (slow dynamics) due to the dissipative nature of parabolic PDEs. Thus, a finite-dimensional control design can be systematically carried out to stabilize the reduced-order system [1,3–8,17]. In a series of the articles, Balas, Curtain and others (see [3] and references therein) presented the formal verification that such type of control is justified for linear and nonlinear dissipative PDEs. In the present paper we design a finite-dimensional Lyapunov-based controller (6) that stabilizes the slow dynamics of the system. The structure of the proposed controller, that consists of series of spatially dependent actuators affected by time-dependent sensors which are spatially weighted-average deviations of state variable from a set point, was suggested in Ref. [20] where inhomogeneous front solutions of reaction–diffusion PDEs were stabilized. Such a control affects only the leading unstable modes. This prevents the spillover effect because this control is in fact diagonal (decentralized) one [18]. This property advantageously distinguishes our controller from others, in which spillover effect may destabilize the higher modes.

To reduce the analysis to a single-variable presentation we capitalize on the property that \( u \) is responding fast and its balance is assumed to be in a steady state: \( \ddot{u} \). Thus, Eq. (3) (with \( D = 0 \)) is transformed to linear ODE: \( V \ddot{u} = -\beta x - d \dot{u} \). Solving this ODE with respect to \( \dot{u} \) and inserting the solution in Eq. (2) with \( f^*\) we reduce the original system (Eqs. (2) and (3) with \( D = 0 \)) to an one-variable integro-differential equation
\[ Lx_t + Vx_z - x_{zz} = f(x) + \dot{u} + \lambda, \] (7)

\[ \ddot{u} = \bar{u}(0)e^{-(d/V)z} - \frac{\beta}{V} \int_{0}^{z} e^{-(d/V)(z-\xi)}x(\xi) \, d\xi, \] (8)

with some feed condition \( \bar{u}(0) \). Here \( \ddot{u}(0) = \bar{u}|_{z=0} \).

2. Structure of infinite-dimensional control

In this section we evaluate the gain coefficient \( k \) that will assure that the Lyapunov’s function \( J(t) \) (Eq. (1)) of the closed-loop system, with infinite-dimensional control (Eqs. (7), (8) and (5)), decays exponentially for all initial conditions on \( x(z, t) \). We set \( x_i = 0 \) for which the system attains a homogeneous solution \( x = 0 \).

Using \( \lambda = -kx \) we can present \( J_t \) as the sum: \( J_t = \langle x, x_t \rangle = \langle x, Le^{-1}(Vx_z - Vx_z + f(x) - kx + \bar{u}) \rangle = Le^{-1}\langle x, x_z \rangle - \langle Vx, x \rangle + \langle x, f(x) \rangle - \langle x, kx \rangle + \langle x, \bar{u} \rangle \). Using the boundary conditions (8) we calculate the first two terms of this sum:
\[ \langle x, x_z \rangle = -Vx(0)^2 - \|x\|_2^2, \] (9)
\[ -\langle x, x_z \rangle = 0.5(V(0)^2 - x(L)^2). \]

So, \( J_t = Le^{-1}(\|x\|_2^2 + 0.5V(0)^2 - x(L)^2) + \langle x, f(x) \rangle - \langle x, kx \rangle + \langle x, \bar{u} \rangle \). In order to evaluate the term \( \langle x, f(x) \rangle \) we apply the inequality: \( \langle x, f(x) \rangle \leq C(x, x) \) that presents an extension of (4) to the infinite-dimensional case [8] (see Appendix for calculating C). Since \( \|x\|_2^2 > 0, x(L)^2 > 0, x(0)^2 > 0 \) and \( \langle x, kx \rangle = 2kJ \) then we obtain an upper estimate for \( J_t \)
\[ J_t < Le^{-1}(2C(0) - k)J + \langle x, \bar{u} \rangle. \] (10)

To evaluate the term \( \langle x, \bar{u} \rangle \) we use (Eq. (3), \( D = 0 \ crea\)
\[ V\bar{u}_t = -\bar{u}z - \beta x \] to express \( x \) via \( \bar{u} \). Then we can calculate \( \langle x, \bar{u} \rangle = (1/\beta)(-\bar{u}z - V\bar{u}_t + \bar{u}) = (d/V)(\bar{u}(0)^2 - \bar{u}(L)^2). \) Since the values \( \bar{u}(0), \bar{u}(L)^2 \) are positive then for \( \bar{u}(0) = 0 \) we find the following inequality for \( J_t \):
\[ J_t < Le^{-1}(2C(0) - k)J. \] (11)

Thus, if \( k > C \) then the equilibrium \( x = 0 \) is globally stable in \([0, L] \) for all parameters and initial conditions \( x(0, t) \). Let us note, if \( \bar{u}(0) \neq 0 \) then (10) becomes

\[ J_t < Le^{-1/2}(2C(0) - k)J. \] (12)

1 Other modes may be touched slightly in realistic feedback control but the dissipative nature of the considered PDE overcomes this problem.
\( J_t < Le^{-1}(2-C)J + V\bar{u}(0)^2/\beta \). \( J_t \) be negative if \( k > C \) and 
\[
\text{abs}(Le^{-1}(2-C)J) > V\bar{u}(0)^2/\beta.
\]

Eq. (10) reveals that a linear feedback (5) with a sufficiently large gain \( k \) assures Lyapunov’s global stability in system (7)–(8). This control is infinite-dimensional and, therefore, cannot be realized in practice. In the next section we design a finite-dimensional analogue of this control. It should be emphasized that we cannot claim that then finite-dimensional control assures global-stability. Obviously, the system will approach global-stability as more terms are added and the domain of attraction is larger with more terms.

3. Finite-dimensional control

To design a realistic finite-dimensional control that stabilizes the slow (unstable) modes we employ a lumped representation of the PDEs model. Because of the dissipative property of the considered PDE its higher modes are stable. In this work we use a spectral decomposition of solutions in terms of the eigenfunctions of the linear operator of (7) since it provides the simplest structure of the lumped system.

We use the Galerkin method for lumping the closed-loop system (Eqs. (7), (8) and (6)) by expanding \( x(z,t) = \sum a_i(t)\phi_i(z) \) in some neighborhood of the homogeneous solution \( x_i = 0 \). Using standard procedures we find an \( n \)-truncated approximation of lumped nonlinear ODEs

\[
Le\bar{a}_i = (-A + Q)\bar{a}_i + \tilde{f}(\bar{a}) + \bar{v} = \bar{F}(\bar{a}), \quad \bar{v} = -k\bar{L}_a\bar{a},
\]

where \( \bar{a} = [a_j(t)], \tilde{f}(\bar{a}) = [f_j(\bar{a})], \bar{F}(\bar{a}) = [F_j(\bar{a})], \bar{v} = [v_j] \) are \( n \)-vectors, \( A = \text{diag}(\mu_1, \ldots, \mu_n), L_n = \text{diag}(1, \ldots, 1) \) and \( Q = [q_{ij}] \) matrices. The control elements \( q_{ij} \) are defined as \( q_{ij} = \langle u(\phi_i), \phi_j \rangle \) where \( u(\phi_i) = -\left(\beta/V \right)\int_0^L f(\bar{z})e^{-(\alpha/V)(z-c^t)}\phi_i(z)\bar{z} \langle \bar{z} \rangle \). The non-linear functions \( f_j(\bar{a}) \) are calculated as follows \( f_j(\bar{a}) = \int_0^L f(\bar{z})\phi_j^0(\bar{z})\bar{z} \langle \bar{z} \rangle d\bar{z} \) for the problem \( \phi_{zz} - V\phi_z = -\mu\phi \) subject to the Danckwerts’ boundary conditions (see [16] for derivation).

The Lyapunov quadratic function (1) for a lumped \( n \)-terms truncated system (11), (12) is transformed as follows:

\[
J^* = \frac{1}{2} \int_0^L \left( \sum_{i=1}^n a_i(\phi_i)^2 \right) d\bar{z} = \bar{a}^T P \bar{a},
\]

where elements of the \( n \times n \) symmetric matrix \( P \) are \( p_{ij} = \langle \phi_i, \phi_j \rangle, \) \( i, j = 1, 2, \ldots \). Moreover, since \( J^* > 0 \) then \( P > 0 \) for any finite \( n \). To simplify further design we propose that \( P = \text{diag}(p_1, p_2, \ldots, p_n) \) with positive \( p_i = \langle \phi_i, \phi_i \rangle \).

Let us compute the time derivative of \( J^* \) for system (11), (12) linearized about \( \bar{a} = 0 \):

\[
J^*_t = Le^{-1}(\bar{a}^T(\bar{F}_a^T P + P \bar{F}_a)\bar{a}).
\]

Here \( \bar{F}_a \) is an \( n \times n \) Jacobian matrix calculated as follows:

\[
\bar{F}_a = \frac{\partial F(\bar{a})}{\partial \bar{a}} \bigg|_{\bar{a}=0} = (-A + Q - kI_n + G(0)),
\]

where the elements of matrix \( G(0) \) are \( g_{ij} = \langle \partial f_i(\bar{a})/\partial a_j \rangle \bar{a} = 0 \), \( i, j = 1, 2, \ldots \).

The value \( J^*_t \) is a negative definite if the matrix \( \bar{F}_a^T P + P \bar{F}_a \) is a negative definite matrix. Thus, we need to find the gain coefficient \( k \) so that the matrix \( \bar{F}_a^T P + P \bar{F}_a \) be negative definite. Further we will repeatedly use the sufficient stability criterion [12] that follows from Gershgorin’s theorem: if an \( n \times n \) matrix \( A \) is a negative dominant diagonal matrix, i.e. its diagonal elements \( \mu_i \) satisfy the criterion

\[
a_{ii} < -\sum_{j=1,j\neq i}^n |a_{ij}|, \quad i = 1, \ldots, n,
\]

in rows or in columns, then all eigenvalues of this matrix have negative real parts.

Let us consider the following useful Lemma.

Lemma 1. If the matrix \( A = [a_{ij}] \) satisfies inequality (15) in rows and the following inequality in columns:

\[
a_{ii} < -\sum_{j=1,j\neq i}^n p_j |a_{ij}|, \quad i = 1, \ldots, n,
\]

then the matrix \( A^TP + PA \) is a negative definite matrix.

Proof. If we multiply (15) and (16) by \( p_i > 0 \) and add the left and right parts of the resulting inequalities then we obtain

\[
2p_i a_{ii} < -\sum_{j=1,j\neq i}^n p_i |a_{ij}| - \sum_{j=1,j\neq i}^n p_j |a_{ij}|, \quad i = 1, \ldots, n.
\]

These inequalities coincide, in fact, with the sufficient stability condition ensured that the symmetric matrix \( A^TP + PA \) is a negative dominant diagonal matrix. Hence, it is a negative definite matrix.

Thus, we need to find a gain coefficient \( k \) so that the matrix \( \bar{F}_a \) satisfies the conditions of the Lemma.

Let us present \( \bar{F}_a = A - kI_n \) where \( A = -A + Q + G(0) \) is the Jacobian matrix of the open-loop system in \( \bar{a} = 0 \) and the term \(-kI_n \) is the effect of control. Because of the dissipative nature of parabolic PDEs, only several \( m_r(m_c) \leq n \) first rows (columns) of the open-loop matrix \( A \) do not satisfy conditions (15) and (16). Hence, to assure the above conditions for the
closed-loop matrix $\tilde{F}_a$ it is enough to shift $m = \max(m_r, m_c)$ of the diagonal elements of $A$ to the left part of the complex plane by control. This operation may be realized by diagonal (or decentralized) feedback control \cite{18} of the form

$$
\tilde{v} = -k (\text{diag}(I_m, O)) \tilde{a}, \quad m = m_r (m_c)
$$

(17)

with a sufficiently large gain $k$ (for details see \cite{20}). Here $\text{diag}(I_m, O)$ is $n \times n$ block-diagonal matrix. In its final form the matrix $\tilde{F}_a$ becomes: $\tilde{F}_a = A - \text{diag}(k I_m, O)$. The value of the gain coefficient $k$ in (17) is calculated as follows:

$$
k \geq \min(k_r, k_c),
$$

(18)

where $k_r, k_c$ satisfy the inequalities

$$
a_{ii} - k_r < - \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad a_{ii} - k_c < - \sum_{j=1, j \neq i}^{n} p_j / p_i |a_{ij}|,
$$

$$
i = 1, \ldots, m.
$$

(19)

Thus, control (17) with $m = \max(m_r, m_c)$ and $k$ from (18) that provides negative definiteness to matrix $\tilde{F}_a P + P \tilde{F}_a$ assures Lyapunov’s stability \cite{19} of the homogeneous solution of finite-dimensional closed-loop system (11), (17) within a neighborhood of $\tilde{a} = 0$. Finite-dimensional control (17) is realized in the original system (Eqs. (2) and (3)) by formula (6).

Remark 1. From above statement it follows that the finite-dimensional control (6) assures Lyapunov’s stability (with criterion (13)) of the $n$-truncation finite-dimensional approximation of the PDE for any finite $n$. Due to the dissipative structure of parabolic PDEs we may expect the achievement of Lyapunov’s stability in the whole infinite-dimensional system for a sufficiently large $n$. In the Appendix this fact is formally proved for the considered kinetics.

Remark 2. We may use any basis functions $\phi_i(z)$ in control (6). For example, to simplify actuator/sensor realization we can apply cosine functions that approximate the eigenfunctions of an unbounded system \cite{15}. The form of finite-dimensional control is not affected by this choice but the values $m$ and $k$ should be recalculated.

**Domain of attraction of the finite-dimensional control**: Analysis of the infinite-dimensional control (5) provides an estimate of the gain ($k > C$) that assures global stability of the PDE model for any initial conditions. The size of the domain of attraction of the finite-dimensional control (which depends simultaneously on $m$ and $k$) can be approximated by a maximal bound $\delta$ on values of states, $|a_i| < \delta$, such that every closed-loop trajectory (that satisfies $J^*(a_i) > 0$) starting in a domain $\Omega = (\mathbb{R}^n, |a_i| < \delta)$ will satisfy condition $J^*(a_i) < 0$. Let us consider the set of matrices calculated in the domain $\Omega$

$$
\hat{F}_a = \frac{\partial G(\tilde{\alpha})}{\partial \tilde{\alpha}} |_{|a_i| < \delta} = -A + Q - \text{diag}(k I_m, O) + \hat{G}(\tilde{\alpha}).
$$

(20)

In (20) $\hat{G}(\tilde{\alpha})$ denotes the set of matrices with elements $\hat{g}_{ij} = (\partial f_j(\tilde{\alpha})/\partial a_i) |_{|a_i| < \delta}$. The envelope of this set may be approximately described by the matrix $G(\delta)$ with elements $g_{ij} = (\partial f_j(\tilde{\alpha})/\partial a_i) |_{a_i = \pm \delta}$. Setting $G(\delta)$ in (20) we can numerically evaluate a maximal $\delta$ that assures inequalities (19) for the set of matrices $\hat{F}_a$ (and hence condition $J^*(a_i) < 0$). For example, for the cubic nonlinearity ($f(x) = -x^3 + x$) we have $g_{ij}(\delta) = \tau_{ij} - 3\delta^2 \int_0^1 (\sum_{j=1}^{n} f_j) \phi_j \phi_i^2 dt$ where $\tau_{ij} = 1$ when $i = j$ and $\tau_{ij} = 0$ otherwise.

**Remark 3.** For most practical applications the shape of the actuator distribution functions in (6) cannot be chosen directly as a set of eigenfunctions $\phi_i$ due to technical constraints. To overcome this problem we approximate $\phi_i$ by the assigned actuator functions $\psi_j$, $i = 1, \ldots, q$ as $\phi_i \simeq \sum_{j=1}^{q} \beta_{ij} \psi_j$ where $\beta_{ij}$ are constants. Then we can design control variables $v_j$, $j = 1, 2, \ldots, m$ in the form $v_j = -k_j \sum_{i=1}^{m} \gamma_{ij} a_i$ where $\gamma_{ij} = \gamma_i (\beta_{ij})$ are some constants and the gain coefficients $k_j$ need to be determined (for details see \cite{20}).

4. Applications

We consider the problem of suppressing patterns (spatially periodic solutions) that emerge in system (2), (3) with cubic kinetics ($f(x) = -x^3 + x$) in the absence of control. Linear

![Fig. 1. Testing controller (17) by applying nonlinear analysis of Eqs. (11) and (12) ($n = 4$, other parameters: $Le = 100$, $d = 0.2$, $\beta = 0.45$, $L = 10$, $V = 1.1$). The figure presents phase portraits in $a_1$, $a_2$, $a_3$ plane obtained for different initial conditions: $a_1(0), a_2(0), a_3(0), a_4(0)$ with $|a_4(0)| = 0.02$: (a) without control ($k = 0$); (b) control with $m = 3$, $k = 0.3$; (c) same with $m = 3$, $k = 1$; (d) same with $m = 2$, $k = 1$; (e) same with $m = 1$, $k = 1$.](image-url)
stability analysis reveals that the homogeneous solution ($x = 0$) is unstable when the bifurcation parameter ($V$) lies within $V_1^* < V < V_2^*$ where $V_1^*$ denotes the linear stability threshold and $V_2^*$ is the absolute-stability threshold for bounded system; $V_2^* \to V_{abs}$ as $L \to \infty$ where $V_{abs}$ is the absolute instability threshold [11] for unbounded system ((2), (3), $D = 0$) (e.g. $V_1^* = 0.9$, $V_2^* = 1.4$ for $L = 10$ and $V_1^* = 0.8$, $V_2^* = 1.5$ for $L = 20$; other parameters are $Re = 100$, $\beta = 0.45$, $d = 0.2$). Self-organized patterns of a certain characteristic wave number, including moving waves and stationary patterns, emerge for $V_1^* < V < V_2^*$. For $V > V_2^*$ the system is convectively unstable, i.e., perturbations grow in a coordinate frame moving with the stream. Here the system is stable to any deviations from the homogeneous solution $x = 0$ but there exist stationary patterns with a large domain of attraction which are induced by constant perturbation imposed at the boundary (i.e. where, $x_{in} \neq 0$, see [11,15] for a detailed analysis).

The number of unstable eigenvalues for $V_1^* < V < V_2^*$ passes through a maximum. A typical phase portrait of an unstable focus of system (11) (with $\tilde{u} = 0$) in the $(a_1, a_2, a_3)$ phase-space is shown in Fig. 1(a) using a 4-term truncated version. Here $n = 4$ is the minimal truncated order which approximates the system of $L = 10$ for all $V \in [0.9, 1.4]$; $n$ increases with growing $L$ ($n = 8$ for $L = 20$).

Analysis of the Jacobian matrix of the open-loop system (11) in $\tilde{u} = 0$ ($A = -A + Q + L\rho$) for the set of parameters: $Le = 100$, $\beta = 0.45$, $d = 0.2$, $L = 10$ reveals that we need to use control (17) with $m = 3$. From inequalities (18), (19) we find that the minimal gain coefficient $k \geq 0.3$ is enough to assures asymptotic stability of the solution $\tilde{u}$ of a closed-loop system (Fig. 1(b)). The larger gain $k \geq 1$, obtained from the analysis of ideal control (Eq. (10)) with $C = 1$ (see Appendix), ensures fast convergence (Fig. 1(c)). Recall that criterion (15) is a sufficient one. Hence, the above estimate of $m$ is an upper bound. Analysis of closed-loop system reveals that control (17) with a smaller $m = 2$ is sufficient for stabilizing (Fig. 1(d)) but $m = 1$ is insufficient to stabilize the system (Fig. 1(e)). The effectiveness of control (6) with two actuators ($m = 2$) and $k = 1$ in original PDEs ((2), (3), $D = 0$) is demonstrated in Fig. 2 in the time domain. Control (6) with a sufficient gain coefficient can effectively suppress patterns in the absolutely $(V_1^* < V < V_2^*)$ and convectively $(V > V_2^*)$ unstable original system ((2), (3), $D = 0$). The simulation of the original system (Eqs. (2), (3), $D = 0$) with control (6) by using finite-difference scheme demonstrates high effectiveness of this control for different values of the parameter $V$ (see Fig. 3(a)–(c)).

To simplify the calculation of the domain attraction of control (17) we consider the domain $\Omega = (R^n, 0 < a_i < \delta)$ where the matrices $\hat{G}(\tilde{a})$ have negative diagonal elements. In this case inequalities (19) may be replaced by the relevant diagonal dominance condition [19] of the matrix $F_0(\delta) = -A + Q - \text{diag}(kI_3, O) + G(\delta)$. Calculating a maximal $\delta$ that assures...
diagonal dominance reveals increasing size of the domain attraction with growing \( k \) (Fig. 4).

5. Concluding remarks

We have developed a methodology to systematically design a finite-dimensional spatially weighted feedback control, that is based on Lyapunov’s direct method. We apply it to stabilize the homogeneous solution of reaction–convection–diffusion systems, in which moving or stationary patterns emerge in the absence of control. We obtain a general structure of control that guarantees stability of the system according to Lyapunov’s direct method. The number of actuators and the gain coefficient affect the domain of attraction of the closed-loop system. The development of infinite-dimensional control exploits the theory of sector nonlinearity. The finite-dimensional realization of this control is based on a spectral representation of the system and dissipative nature of parabolic PDEs. This control ensures negative diagonal dominance in rows and columns of the Jacobian matrix of the system calculated in a neighborhood of the homogeneous solution. We propose a construction methodology for estimating stability region of the closed-loop system. We remark that we considered only boundary conditions for estimating stability region of the closed-loop system.

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Appendix

Calculation of \( C \). Consider the rational function: \( f(x) = b_0 x^n + b_{n-1} x^{n-1} + \cdots + b_1 x \) with real coefficients \( (b_j < 0) \) and odd highest degree \( n \). It is evident that \( f(x) \) satisfies the condition \( f(x) \to -\infty \) as \( x \to \infty \) (\( f(x) \to \infty \) as \( x \to -\infty \)). So, \( f(x) \) is bounded from above for \( 0 \leq x \leq \infty \) (\( f(x) \) is bounded from below for \(-\infty \leq x \leq 0\)). Consider the case of \( f(x) \) with real roots. Sector condition (4) geometrically implies that the nonlinear function \( f(x) \) satisfies the inequalities: \( f(x) \leq C x \) for \( x \geq 0 \) and \( f(x) \geq C x \) for \( x < 0 \). An upper estimate \( (C_m) \) of the \( C \) value can be obtained as a maximum of the function \( f(x) = f(x)/x \). For example, for \( f(x) = -x^3 + x \) we have \( f(x) = -x^2 + 1 \) and \( C_m = 1 \). For the quintic nonlinearity \( f(x) = -x(x^2-1)(x^2-a^2) \) we obtain that \( f(x) = -4x^2 - 2(1+a^2)x \) has three roots: \( x_1 = 0 \) and \( x_{2,3} = \pm \sqrt{0.5(1+a^2)} \) which we use to calculate \( C_m = 0.25(1-a^2)^2 \).

Proof of Remark 1. Let us extend Lyapunov’s quadratic function (13) to the infinite-dimensional case as

\[
J = J^* + J^{in},
\]

where \( J^* \) is defined from (13) and \( J^{in} = (1/2) \int_0^\infty \sum_{i=1}^n a_i^2 \phi_i^2 \) \( dz = \bar{a}_i^T P_m \bar{a}_i \). Here \( P_m = diag(p_{n+1}, \ldots) \) is infinite-dimensional matrix with \( p_i = \langle \phi_i, \phi_i \rangle, i = n + 1, n + 2, \ldots \).

Calculating the time derivation of \( J \) for infinite-dimensional system (11), (12) linearized about \( \bar{a} = 0 \) gives

\[
J_t = J^*_t + J^{in}_t = L e^{-1} \left[ \bar{a}^T (\tilde{F}^P_a + \bar{F}_a) \bar{a} + \bar{a}_i^T (\tilde{F}^{in}_a + F_m) P_m \bar{a}_i \right].
\]

Here \( \tilde{F}^{in}_a = (-A_{in} + Q_{in} + G_{in}(0)) \) is an infinite-dimensional matrix with \( A_{in} = \text{diag}(\mu_{n+1}, \ldots), Q_{in} = [q_{ij}], q_{ij} = (u(\phi_i), \phi_j^T), G_{in}(0) = [g_{ij}], g_{ij} = (\bar{f}_i(\bar{a})/\bar{a}_i) \), i.e., \( i = n + 1, n + 2, \ldots \).

The value \( J_t \) is a negative one if the matrices \( (\tilde{F}^{in}_a + P_m \tilde{F}_a) \) and \( ((\tilde{F}^{in}_a)^T P_m + P_m \tilde{F}_a) \) are negative definite matrices. Let us show that for the infinite-dimensional system (11) there always exists a finite \( n \) such that matrix \( ((\tilde{F}^{in}_a)^T P_m + P_m \tilde{F}_a) \) has strictly negative eigenvalues (or it is a negative definite matrix).

Indeed, the matrix \( \tilde{F}^{in}_a \) with polynomial kinetics \( f(x) \) (see above) has the form \( \tilde{F}^{in}_a = -A + Q + G(0) \) where \( A = \text{diag}(\mu_{n+1}, \mu_{n+2}, \ldots), Q(0) = \text{diag}(b_1, b_1, \ldots) \) and \( Q = [q_{ij}] \), i.e., \( i = n + 1, \ldots, j = n + 1, \ldots \) where \( q_{ij} = \langle -\beta V y \rangle \int_0^L \phi_j^a(z) \int_0^z e^{-d/V z} \phi_i(z) dz \). Evaluating elements of the matrix \( Q \) reveals that diagonal elements \( q_{ii} \) are finite values and off-diagonal elements \( q_{ij}, i \neq j \) tend to zero with growth of \( i \) and \( j \). By virtue of the fact that inequalities (15) were introduced only for finite-dimensional matrices, it cannot be used for infinite-dimensional matrices. On the other hand, the infinite-dimensional matrix \( \tilde{F}^{in}_a \) has a finite number \( (v) \) of nonzero off-diagonal elements in every rows. Hence we can apply inequality (15) to \( v \) elements of the first row of \( \tilde{F}^{in}_a \), \( -p_{n+1} + b_1 + q_{n+1,n+1} < - \sum_{i=n+2} v q_{n+1,i} \). Since \( \mu_i, i = 1, 2, \ldots \) grows infinitely with \( i \) there then always exist

\[6\] The similar approach has been used in our work [20].
a finite integer \( i = n + 1 \) such that the above inequality is satisfied. A similar way may be used for inequality (16). Hence every row of the symmetric matrix \((\tilde{F}_a^{\text{in}})^T P_{\text{in}} + P_{\text{in}} \tilde{F}_a^{\text{in}}\) satisfies the sufficient stability condition. Thus, it is a negative dominant diagonal matrix for this \( n \) and has all its eigenvalue are strictly negative. Hence \( J_{\text{in}}^* < 0 \) and control (17) that provides negative definiteness to matrix \( \tilde{F}_a^T P + P \tilde{F}_a \) guarantees simultaneously \( J_t^* < 0 \) and whole \( J_t < 0 \).

Let us note that an approximate estimate of \( n \) may be obtained by conventional methods, for example, by analysis of convergence of leading eigenvalues of \( \tilde{F}_a \) (with \( k = 0 \)).

References