Deformation and breakup of a non-Newtonian slender drop in an extensional flow

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Abstract

The deformation and breakup of a non-Newtonian slender drop in a Newtonian liquid in a simple extensional and creeping flow has been theoretically studied. The power-law was chosen for the fluid inside the drop, and the deformation of the drop is described by a single ordinary differential equation, which was numerically solved. Asymptotic analytical expressions for the local radius were derived near the center and close to the end of the drop. The results for the shape of the drop and the breakup criterion are presented as a function of the capillary number, the viscosity ratio and type of non-Newtonian fluid inside the drop. An approximate analytical solution is also suggested which is in good agreement with the numerical results.

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1. Introduction

When a drop of one fluid is suspended in another fluid undergoing shear or extensional flow, the drop will deform. If the strength of the flow exceeds some critical value, the drop will break into two or more fragments. This phenomenon of deformation and breakup of drops plays a major role in many physical processes. For example, in the rheology of emulsions, where one fluid is dispersed into another, in the design of efficient mixing devices, and in mass transfer in such systems.

The subject of deformation and breakup of a drop suspended in another fluid was first studied by Taylor [1,2]. By extending the theory of Einstein [3] for a solid sphere, Taylor obtained an analytical solution which describes the small deformation of a Newtonian drop suspended in another Newtonian fluid undergoing shear or extensional flow under creeping flow conditions. Since then, this subject has been considered by many authors, the theory was redefined and many experimental results were reported. A summary of these studies can be found in two excellent reviews by Rallison [4] and Stone [5].

The deformation of a non-buoyant Newtonian drop in a Newtonian liquid under creeping flow conditions is governed by two dimensionless numbers. The first one is the capillary number, \( Ca \), the ratio of the external viscous force (which tends to deform the drop) to the surface tension force (which tends to keep the drop spherical). The second one is \( \lambda \), the ratio of the viscosity of the drop to the viscosity of the external fluid. In general, the common practice is to distinguish between small drop deformations and large drop deformations. For small deformations, \( Cu \ll 1 \), the drop can be considered a slightly perturbed sphere, and an approximate solution of the Stokes equations can be obtained by a regular perturbation technique. On the other hand, slender drops are obtained in creeping flows when \( Cu \to \infty \) and \( \lambda \to 0 \). The first one to suggest a slender body theory for a drop in Stokes flow was, again, Taylor [6], who studied the deformation of a drop in a simple extensional flow. His solution shows that the slender drop has a parabolic shape with pointed ends. The theory was later refined by Buckmaster [7,8], Acrivos and Lo [9] and others to include different effects as well as other types of flows.
Recently, Favelukis and Nir [10] extended the slender body theory to consider the deformation of a bubble (an inviscid drop) in a non-Newtonian liquid in a simple extensional flow. The majority of the work presented in the literature is restricted to small drop deformations in Newtonian systems. This is far from reality in many industrial applications, such as polymer blends, where large deformations can be present in systems, which exhibit non-Newtonian behavior such as shear thinning and elasticity. It is, therefore, not surprising that lately many researches are being focused on the deformation problem of a non-Newtonian drop in an extensional creeping flow. Experimental studies report, typically, small to medium deformations of visco-elastic drops [11–15] with a time dependent rheology. In these studies transient effects play an important role and quantitative measures for critical large deformations are not easily deduced. Few such experiments with large deformations of drops suspended in visco-elastic media are also reported. For example, slender drops were observed in the entrance region to a tube [16], in planar extensional flow [17] or in uniaxial elongational flows [18]. Numerical simulations using boundary integral formulations [19,20] or boundary elements methods [21–23], as well as various asymptotic analyses [24–26] also cover mostly small or medium deformations of non-Newtonian drops with a variety of rheological characteristics. Thus, critical criteria for breakup are not generalized. Furthermore, there is no study that is primarily focusing on large deformations and slender drops. Nevertheless, one common basic conclusion can be drawn from these studies. They indicate that low viscosity non-Newtonian drops are less deformable than Newtonian droplets (see, e.g [17]). It can be, therefore, anticipated that critical lengths of such highly deformed drops are shorter than that of the slender Newtonian drop.

It is the purpose of this paper to present an analysis for the steady deformation and for the breakup of a non-buoyant, non-Newtonian low viscosity slender drop in a Newtonian liquid in a simple extensional and creeping flow. In this study, we take into consideration the fact that the flow within most of the volume of the slender drop is dominated by shear components. Thus, simplified rheological models can be employed. We then start this work by presenting the governing equations for both the flow inside and outside the drop, where the simple power-law model is assumed for the non-Newtonian fluid inside the drop. Next, the solutions relating the dynamics of a bubble (an inviscid drop) and a Newtonian drop, as published in the literature, are revisited. The deformation of a non-Newtonian slender drop in a Newtonian liquid in a simple extensional flow then follows. Exact numerical results and approximated analytical calculations for the shape of the drop as well as the breakup criterion are presented as a function of the capillary number, the viscosity ratio and the type of non-Newtonian fluid. Finally, the possibility of comparison with previous experimental results is discussed.

2. The governing equations

2.1. The flow outside the drop

A simple (axisymmetric) extensional flow is defined in cylindrical co-ordinates by the following velocity compo-
The flow inside the drop is a Poiseuille flow (Taylor [6]), that the normal component of the surface velocity at the boundary. If inertia forces are neglected, the velocity profile, Eq. (6), may be integrated to give the volumetric flow rate inside the drop:

\[ Q = \int_{0}^{z} \frac{G_{z}}{\pi R^{2}} \left( \frac{n}{1+n} \right) n G_{z}^{n} \frac{z^n}{R^{1+n}} \right] . \quad (7) \]

An expression for the axial pressure gradient may be obtained by the condition that the volumetric flow rate must vanish at every cross-section inside the drop:

\[ \frac{dP}{dz} = 2 \frac{n}{1+n} n G_{z}^{n} \frac{z^n}{R^{1+n}} . \quad (8) \]

The integration of the last equation results in:

\[ P = P(0) + 2 \frac{n}{1+n} n G_{z}^{n} \int_{0}^{z} \frac{z^n}{R^{1+n}} dz . \quad (9) \]

where

\[ u(n) = \left( \frac{1+n}{n} \right)^{n} . \quad (10) \]

Here \( P(0) \) is the unknown pressure at the center of the drop. For an incompressible bubble (an inviscid drop), \( m = 0 \), and the pressure inside the bubble is constant.

Upon substitution of the pressure gradient, given by Eq. (8), into Eq. (6) we obtain the complete expression for the velocity profile:

\[ \frac{v_{z}}{G_{z}} = 1 - \frac{1+n}{1+n} \left[ 1 - \left( \frac{z}{R} \right)^{1+n} \right] . \quad (11) \]
A normal-stress balance at the surface of the slender drop can be written as:

\[ P - P_{\text{ext}} - \tau_{\text{in}} + \tau_{\text{out}} = \frac{\sigma}{R} \]  

(12)

where \( P \) is the variable pressure inside the drop and the subscripts in and out denote the regions inside and outside the drop, respectively. Slender drops are encountered if the viscosity of the drop is much smaller than the viscosity of the liquid. Since, the velocity gradients inside and outside the drop are of the same order of magnitude, it follows that the normal viscous stress inside the drop can be neglected. Also, for the Newtonian liquid outside the drop the pressure of the disturbed flow is, at first approximation, also constant and the pressure of the undisturbed flow is inside the drop. Also, for the Newtonian liquid, the normal viscous stress in the liquid outside the drop can be written as:

\[ \tau_{\text{nn}} = -2\mu \frac{\partial v_z}{\partial r} \]

(13)

where \( \mu \) is the viscosity and \( v_z \) is the velocity component. Note that, for simplicity, the in and out subscripts are omitted.

The normal viscous stress of the Newtonian liquid (viscosity \( \mu \)), outside the drop at the drop surface, can be obtained with the help of Eq. (13) to give:

\[ (\tau_{\text{nn}})_{z=R} = 2\mu \left( \frac{\partial v_z}{\partial r} \right)_{z=R} = -2\mu G \left( 1 + \frac{z}{R} \frac{dR}{dz} \right). \]

(14)

substituting Eqs. (9) and (14) into Eq. (13) results in:

\[ P(0) + 2\mu m G^2 \int_0^z \frac{dz'}{R^{1+n}} = -\mu G \left( 1 + \frac{z}{R} \frac{dR}{dz} \right) \frac{\sigma}{R} \]

(15)

In dimensionless form, the equation reads

\[ P^*(0) + 2\mu \lambda \int_0^z \frac{dz'}{R^{1+n}} = \frac{1}{CaR^*} \]

(16)

where the pressure and the normal-stress are rendered dimensionless with respect to the characteristic stress outside the drop (\( \mu G \)) and all the lengths with respect to \( R \) (the radius of a sphere of equal volume).

Here, the viscosity ratio is given by:

\[ \lambda = \frac{mG^{-1}}{\mu} \]

(17)

and the capillary number is defined as:

\[ Ca = \frac{\mu Ga}{\sigma} \]

(18)

The capillary number represents the ratio of the external viscous force, which tends to deform the drop to the surface tension force which tends to keep the drop spherical.

Note that at the center, \( z = 0 \), the radius of the drop is related to the unknown internal pressure there. The latter can be eliminated if Eq. (16) is differentiated with respect to \( z \).

After some algebraic manipulation we obtain:

\[ 2\lambda \int_0^z \frac{dz'}{R^{1+n}} = \frac{1}{CaR^*} \]

(19)

and the shape of the drop can be obtained by solving the last equation with the following two boundary conditions:

\[ R^*(0) = \frac{1}{2\lambda Ca} \]

(20)

\[ R^*(L^*) = 0. \]

(21)

where the dimensionless parameter \( v \), the rescaled pressure, is defined as:

\[ v = \frac{1}{2} P^*(0) - 1. \]

(22)

Finally, the half-length of the drop is obtained by assuming that the volume of the drop is conserved:

\[ \int_0^{L^*} R^{2n} dz^* = \frac{2}{3} \]

(23)

From Eqs. (13) and (14), one can see that the normal-stress at the surface of the drop exerted by the external fluid of \( O(\mu G) \) must be balanced by the capillary force of \( O(\sigma / R) \) leading to \( R = O(1/Ca) \). Hence, the conservation of volume,
given by Eq. (23), suggests that $L^* = O(\nu^{3/4})$. If a small parameter $\varepsilon = R^*/L^* = O(1/\nu^{1/4})$ is defined then from Eq. (19) we obtain that $\varepsilon^{1/4} = O(\lambda)$ or $CaL^{3/4}(\nu^{1/4}) = O(1)$ while from the volume conservation we also have $L^*/\lambda^{3/4} = O(1)$.

It follows from this order of magnitude consideration that the rescaled length $g = L^*/\lambda^{3/4}$ must be a function of the rescaled capillary number $f = CaL^{3/4}(\nu^{1/4})$.

Another parameter, which is useful in mass transfer problems, is the dimensionless surface area defined at first approximation as:

$$A^* = \frac{A}{4\pi a^2} = \int_0^{L^*} (z^*)^2 \, dz^*,$$

where $A$ is the surface area of the drop.

Before starting to solve the problem, the solution relating the dynamics of a bubble (an inviscid drop) and a Newtonian drop will be revisited (see Taylor [6], Buckmaster [7,8], and Acrivos and Lo [9]).

3. A bubble in a Newtonian liquid

For an incompressible bubble (an inviscid drop), $\lambda = 0$ and the pressure inside the bubble is constant. Combining Eqs. (16) and (22) results in:

$$\frac{dR^*}{dz^*} - vR^* = -\frac{1}{2C\sigma},$$

The solution of Eq. (25) and the boundary condition given by Eq. (21) results in:

$$R^* = \frac{1}{2C\sigma} \left(1 - \left(\frac{z^*}{L^*}\right)^2\right).$$

Substituting the last equation into Eqs. (23) and (24) gives:

$$L^* = \frac{4}{3}(v + 1)(2v + 1)Ca^2,$$

$$A^* = \frac{2}{3}(2v + 1)Ca.$$

According to Taylor’s solution [6], $v = 2$. Buckmaster [7] argued that in order to obtain an analytical shape the rescaled pressure $v$ must be an even integer and he made the same choice as Taylor did, at which the smallest possible deformation is obtained. Acrivos and Lo [9] arrived at the same conclusion. By solving the singular problem in the region $z^* \to 0$, they showed that only $v = 2$ results in a stable shape.

It follows from Eq. (22) that the pressure in the bubble is $P^* = 6$ and that the bubble deformation parameters are:

$$R^* = \frac{1}{2C\sigma} \left(1 - \left(\frac{z^*}{L^*}\right)^2\right),$$

$$L^* = 20Ca^2,$$

$$A^* = \frac{20}{3}Ca.$$

The theory presented here for the deformation of a slender bubble in a Newtonian liquid was recently extended to consider the case in which the liquid outside the bubble is non-Newtonian (Favelukis and Nir [10]).

4. A Newtonian drop in a Newtonian liquid

For a Newtonian drop in a Newtonian liquid, $n = 1$, $\nu(1) = 4$ and Eq. (19) reduces to:

$$2\nu \frac{d^2R^*}{dz^*} - 2\nu \left(\frac{dR^*}{dz^*}\right)^2 + \left(2 - \frac{1}{CaR^*}\right) \left(\frac{dR^*}{dz^*}\right) = 8\nu \frac{z^*}{R^*}.$$

The solution of Eq. (32) together with the boundary conditions given by Eqs. (20) and (21), results in:

$$R^* = \frac{1}{2C\sigma} \left(1 - \left(\frac{z^*}{L^*}\right)^2\right),$$

where the rescaled pressure $v$ satisfies:

$$8\nu \frac{z^*}{R^*} - v + 2 = 0,$$

and:

$$K^2 = \frac{1}{C\sigma L^2\lambda} = \frac{1}{f^2g^2}.$$ 

For the Newtonian drop, $f = CaL^{3/4}$ and $g = L^*/\lambda^{3/4}$ and the solution for $v$ is:

$$v = \frac{4}{1 \pm \sqrt{1 - 64K^2}}.$$ 

It follows that the half-length of the drop is of the form:

$$L^* = \frac{5\nu^2}{2C\sigma},$$ 

and the dimensionless surface area of the drop is:

$$A^* = \frac{5}{3}Ca.$$ 

Finally, Eqs. (34), (35) and (37) are combined to obtain the deformation curve (see Fig. 3 for $n = 1$):

$$f = \frac{1}{\sqrt{20}(1 + (4/5)g^2)}.$$

Acrivos and Lo [9] argued that a stable solution is obtained at the lower part of the curve where $2 \leq v \leq 2.4$, while the upper section where $2.4 \leq v \leq \infty$ represents an unstable solution. The burst of the drop occurs at $f = f_{\text{max}}$ ($K^2 = 576/5$ or $v = 2.4$), therefore, the breakup criterion is:

$$f_{\text{max}} = CaL^{3/4} = 0.148.$$ 

For the specific case of a bubble (an inviscid drop) in a Newtonian liquid: $\lambda = 0$, $K = \infty$, and $v = 2$, see Section 3.
that according to Eq. (40) a bubble, contrary to a drop, does not break.

5. A non-Newtonian drop in a Newtonian liquid

In order to simplify the mathematical presentation, we define the following rescaled variables: \( y = R \frac{\rho}{C_2} \) and \( x = \frac{\xi}{C_2} \).

Note that, according to our discussion given in the paragraph below Eq. (23), both variables are of the order of magnitude of 1. With the new definitions, the problem to be solved in Eqs. (19)–(21), (23) and (24) is transformed into:

\[
2x^2 \frac{d^2 y}{dx^2} - 2 \frac{d y}{dx} \left( 2 - \frac{1}{\gamma} \right) \frac{d y}{dx} = 2w f^{3n+1} \left( \frac{x}{y} \right)^{n},
\]

(41)

\( y(0) = \frac{1}{2\nu}, \)

(42)

\( y(x_L) = 0, \)

(43)

\[
\int_{0}^{x_L} y^2 dx = \frac{2}{3}, \quad \frac{A^*}{C_a} = \int_{0}^{x_L} y dx,
\]

(44)

where \( x_L = L'/C_a^2 \), and with \( f = C_2^2 \alpha \beta \gamma \). Note that the above definitions simplify the problem since the governing equation is now a function of two parameters (\( \alpha, \beta \)) instead of three (\( n, C_a, \lambda \)). Note also that the rescaled pressure \( \frac{\rho}{\nu} \) is still to be determined.

5.1. Asymptotic expansion near the center of the drop

At the center of the deformed drop the rescaled radius is \( y(0) \). We define the deviation from this value by:

\[
\rho(x) = y(0) - y(x),
\]

(46)

where it is assumed that near the center \( \rho(x) \ll y(0) \). Substituting the above definition into Eq. (41) and deleting terms that are of \( O(\rho^4) \) or higher we obtain the following second order differential equation:

\[
-x^2 \frac{d^2 \rho}{dx^2} + (\nu - 1) \frac{d \rho}{dx} = 2w f^{3n+1} \rho^n,
\]

(47)

which may be solved under the condition that \( \rho(0) = 0 \) to give:

\[
\rho = 2w f^{3n+1} \frac{1}{1 + n} \left( \frac{x}{y} \right)^{1+n} + C x^{n-1},
\]

(48)

where \( C \) is an integration constant. With this representation the slope of the leading terms at \( x \to 0 \) becomes:

\[
\frac{dy}{dx} = \frac{2w f^{3n+1} \nu^{n+1}}{\nu - 1 - n} - C x^{n-1}.
\]

(49)

It is clear that both terms in Eq. (48) are not analytic at \( x = 0 \). Hence, there must be an inner singular region near the stagnation point at \( x = 0 \) where the flow is not uni-directional to a first approximation, as was assumed in deriving Eq. (16) or (19), and in which these equations no longer present an appropriate model. A detailed solution at the stagnation region can provide the necessary analytic continuity at \( x = 0 \) and also help selecting the stable solutions among the possible many solutions that may emerge from the system given by Eqs. (19)–(21) (see, e.g. Acivos and Lo [9], for the Newtonian drop case). The analytic inner region should also serve to select the leading term in the asymptotic form (48). Extrapolating the findings in the Newtonian case [9] we assume \( n > n+1 \), in which case the higher order homogeneous term can be neglected to a first approximation and the resulting slope in Eq. (49) remains finite and negative. Nevertheless, since the flow in the inner stagnation region is not expected to affect the deformation pattern or the breakup criteria (that are governed by the outer problem) we leave the analysis there outside the scope of this work.

5.2. The region close to the end of the drop

Near the end of the drop (\( x \to x_L \)), the following profile is assumed:

\[
y = -B(x_L - x), \quad a \geq 0.
\]

(50)

The possible cases \( a > 1, a = 1 \) and \( a < 1 \) correspond to a cusp-like edge, a pointed end with a constant slop and a blunt edge, respectively. Substituting this expression into Eq. (41) we find that no solution that satisfies \( y(x_L) = 0 \) exists if \( n > 1 \). For the cases \( n \leq 1 \), there are two possibilities, \( a = 1 \) and \( a = 1/n \).
Thus, for \( n < 1 \), the leading order is \( \alpha = 1 \) at which the balance reduces to:

\[
-2 \frac{x}{y} \frac{dy}{dx} - \frac{1}{y} \frac{dy}{dx} = 0.
\]

(51)

This last equation together with Eq. (43) can be easily solved to give the following radius profile near the end of the drop:

\[
y = \frac{1}{2} \ln \frac{x}{x_0} = \frac{1}{2} \left( 1 - \frac{x}{x_0} \right).
\]

(52)

Eq. (51) yields drops with pointed ends with a finite and non-zero constant end slope:

\[
\frac{dy}{dx} = -\frac{1}{2x_0}.
\]

(53)

Note that in this approximation the inner viscous effect is completely absent and, thus, the non-Newtonian character of the drop is not directly present at the tip. Hence, although the total deformation depends on the values of \( \alpha \) and the inner viscosity, no dramatic variations of the end shapes are expected with the change of these parameters. It becomes evident below that the asymptotic prediction of non-zero slope expected with the change of these parameters.

5.3. Numerical solution

The non-linear two point boundary value problem can be solved by using a shooting method in which the solution is obtained by using a shooting method in which the solution is obtained by marching with Eq. (41) from \( x = 0 \), with Eqs. (42) and (49) serving as initial values, until \( y(x) = 0 \) is obtained. The correct \( x_0 \) is the one at which the conservation of volume (Eq. (44)) is satisfied. This is a two-dimensional search in the space of the rescaled pressure \( v \) and the rescaled capillary number \( f \). The search can be accelerated by employing the transformation:

\[
\phi = f^{(n+1)} \phi^{n+1},
\]

(54)

by which Eqs. (41)-(43) reduce to:

\[
2(n+1)^2 \frac{d^2 \phi}{d \phi^2} - 2(n+1)^2 \frac{d \phi}{d \phi} \left( \frac{1}{y} \frac{dy}{d \phi} \right) + (n+1) \left( 2n + 2 - \frac{1}{y} \right) \frac{dy}{d \phi} = 2 \frac{v}{v^0},
\]

(55)

\[
y(0) = 0;
\]

(56)

\[
y(\phi_0) = 0,
\]

(57)

and the slope at \( \phi = 0 \) becomes:

\[
\frac{dy}{d \phi} = -\frac{2}{(n+1)(v - 1 - n)}.
\]

(58)

Note that this substitution eliminates the rescaled capillary number \( f \) from the governing equation. It follows that for a choice of a rescaled pressure \( v \) the solution is marched until the value of \( \phi_0 \) where \( y = 0 \) (Eq. (57)) is obtained and then the value of \( f \) is calculated directly using the volume conservation. Thus, the combinations of Eqs. (44) and (55) yields:

\[
f = \frac{3}{2(n+1)} \int_{y=1}^{y_0} \frac{dy}{y \phi^{-(n+1)}},
\]

(59)

and the deformation curve can be obtained by plotting \( g \) as a function of \( f \), wherein

\[
g = f^2 x_L = \frac{1}{\phi_0^{(n+1)}}.
\]

(60)

Finally, the auxiliary result of the surface area of the drop can be obtained from Eqs. (45) and (54):

\[
A^* = \frac{1}{n} \int_{y=0}^{y_0} \frac{dy}{\phi^n}.
\]

(61)

Clearly, the mathematical procedure obtained by the transformation given in Eq. (54) allows us to obtain a numerical solution where the only guess is of the rescaled pressure \( v \). However, it appears that there are difficulties in matching with the numerical calculation directly from \( \phi = 0 \). The first one stems from the restriction \( n < 1 \) which, in view of Eq. (49), implies that \( d^2 y / d \phi^2 \) diverges at the origin. It is also apparent that not every choice of the value of \( v \) results in a numerically calculable solution. Naturally, \( v = 1 + \alpha \) is one such choice and there appear to exist other such singular values for \( v \).

The identification of these cases and the numerical procedure itself are facilitated by expressing the solution of Eq. (55) as a power series in \( \phi \):

\[
y(\phi) = \sum_{\ell=0}^{\infty} b_\ell \phi^\ell,
\]

(62)

where, in view of (56) and (58), the first few coefficients are:

\[
b_0 = \frac{2}{n v(1+n)};
\]

\[
b_1 = \frac{2(1+n)(n+1)}{(1+v)(1+n+v)};
\]

\[
b_2 = \frac{2(1+n)(n+1)^2 v}{(1+n)[v-(1+n)]^2[1-v]};
\]

(63)

The higher order coefficients of Eq. (62) are obtained as recurrence relations by satisfying Eq. (55) for each power of \( \phi \). Note that \( v=1+\alpha \), with \( j = 1, 2, 3, \ldots \), are all singular branch points and we can anticipate that the domain of solutions contains several branches depending on the values of the rescaled pressure \( v \).

The numerical procedure that was finally adopted is as follows: for a choice of \( v \) the value of \( y(\phi) \) and its first derivative were calculated at a non-zero yet small \( \phi \) using the expansion, given by Eq. (62), to a desired degree of accuracy. The solution was then marched numerically to \( y = 0 \) to determine \( \phi_0 \). It should be noted that, since a considerable number of terms
were employed in Eq. (62) (\(N\) being typically between 11 and 30), the calculated value of \(\phi_L\) corresponds also to the first real zero of the polynomial:

\[
\sum_{k=0}^{N} b_k \phi_k^L = 0.
\]

(64)

Thus, for all practical purposes, the expression given by Eq. (62) may replace the numerical solution for the entire range \(0 \leq \phi \leq \phi_L\) provided that a sufficient number of terms are employed.

5.4. Numerical results

The solid curves in Fig. 3 depict the numerical solution for various values of the power-law parameter \(n\). The solution is constructed starting at a low value of \(\nu\) and spanning the range \((n+1) < \nu < 2(n+1)\). The deformation curve of the rescaled length \(g\) versus the rescaled capillary number \(f\) has a form of a lobe reminiscent of the form obtained by Acrivos and Lo [9] when they considered the additional non-linear effect of weak inertia. The lobes present two branches of the solution separated by the point at which \(f\) is maximum \(f_{\text{max}}\). Along the lower branch the deformation is stable while the upper branch represents an unstable configuration. The point of maximum deformation at \(f_{\text{max}}\) is conceived as the critical point beyond which the drop disintegrates. Hence, we conclude that the breakup criterion of a non-Newtonian power-law slender drop in a Newtonian liquid in an extensional creeping flow is given by:

\[
Ca\lambda_1/\left[3(1+n)\right] = f_{\text{max}}.
\]

(65)

The results show that as \(n\) increases \(f_{\text{max}}\) decreases. A summary of some numerical results, at the breakup point, is listed in Table 1.

Fig. 4 shows the half-length of the drop \((x_0 = L^*/Ca^2)\) as a function of \(\nu\) for various values of \(n\). Also shown in this figure is the solution of the governing equation for inviscid drops, given by Eq. (27), represented by the dashed line. Note that on the dashed line the Newtonian bubble is the point \(\nu = 2\), \(x_0 = 20\). The solid circle symbol is placed at the minimum value of \(L^*/Ca^2\), and the squares mark the breakup point. The dimensionless surface area of the deforming drop is given in Fig. 5 with the limiting contour of inviscid drops given by Eq. (28).

We have already indicated that, in view of the expansion of Eq. (62) and the form of the coefficients in Eq. (63), the domain of \(\nu\) contains a series of singular points. Thus, in view of the non-linearity of the governing equation, several branches of the solution can be anticipated. Indeed, such multiplicity is shown in Fig. 6 for the case \(n = 0.5\) in the form of separated lobes. Each additional lobe corresponds to a higher continuous interval of \(\nu\) within the hierarchy of singular points. It is assumed that, except for the lower branch of the lower lobe, all the solutions are physically unstable and cannot be realized. The proof to this assumption should follow the guidelines given by the stability analysis of Acrivos and Lo [9] for the multiple solutions for a Newtonian drop and it is not included in the scope of this paper. It is interesting to note that the solution for \(n < 1\) contains multiple lobes, the solution for \(n = 1\) contains a single lobe (Acrivos and Lo [9]) and the case

\[
\lambda_1 = f_{\text{max}}.
\]

Fig. 4. The half-length of the drop as a function of the rescaled pressure \(\nu\), for different values of \(n\). The solid line represents the drop and the dashed line is an inviscid drop according to Eq. (27). The circles are placed at the minimum value of \(L^*/Ca^2\) and the squares mark the breakup point.

Table 1

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\nu)</th>
<th>(f)</th>
<th>(g)</th>
<th>(L^*/Ca^2)</th>
<th>(A^*/Ca)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.64</td>
<td>0.182</td>
<td>0.152</td>
<td>16.7</td>
<td>2.99</td>
</tr>
<tr>
<td>0.5</td>
<td>1.90</td>
<td>0.171</td>
<td>0.194</td>
<td>20.4</td>
<td>3.33</td>
</tr>
<tr>
<td>0.75</td>
<td>2.15</td>
<td>0.159</td>
<td>0.616</td>
<td>24.4</td>
<td>3.67</td>
</tr>
<tr>
<td>1</td>
<td>2.4</td>
<td>0.148</td>
<td>0.630</td>
<td>28.8</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 5. The surface area of the drop as a function of the rescaled pressure \(\nu\), for different values of \(n\). The solid line represents the drop and the dashed line is an inviscid drop according to Eq. (28). The circles are placed at the minimum value of \(A^*/Ca\) and the squares mark the breakup point.
Fig. 6. Multiple lobes in the deformation curve for the case $n = 0.5$.

$n > 1$ reveals no solution. Furthermore, the case $n = 1$ is not a smooth continuation of the cases at $n < 1$ but, rather, a special case. In the former case the deformation curve is a continuous contour enveloping the entire spectrum of discrete lobes that exist as long as $n < 1$. This is clearly shown in Fig. 7 where we have plotted our results for $n = 0.99$ and the solution of Acrivos and Lo [9] for $n = 1$.

5.5. Some practical approximations

A practical estimate of several physical parameters such as drop length, surface area and the criterion for breakup can be obtained by considering only the first two terms in the expansion of Eq. (62) without resorting to cumbersome numerical calculations. Applying these terms with the boundary conditions, given by Eqs. (56) and (57), suggests:

$$
\phi_L \approx \frac{(1 + n)(v - n - 1)}{2^{n+1}n^{n+1}w}.
$$

(66)

The half-length of the drop becomes:

$$
x_L = \frac{L^*}{\text{Ca}^{\nu}} \approx \frac{4 (2 + n)(3 + 2n)}{3 (1 + n)^{\nu}},
$$

(68)

while the surface area of the drop can be obtained by evaluating Eq. (61) in this approximation to give:

$$
A^* \approx \frac{2 (3 + 2n)}{3 (1 + n)^{\nu}}.
$$

(69)

We have calculated the deformation curves following these estimates, using Eqs. (60), (66) and (67) and the results are shown in Fig. 3 in the form of dashed lines. Clearly, the results show that the $O(\phi)$ approximation, that can be readily calculated, is in excellent agreement with the exact numerical results up to the breakup point. It should be born in mind that for the Newtonian drop case, where $n = 1$, the $O(\phi)$ result is actually the exact result.

The drop breakup point (the maximum value of $f$) can be easily evaluated by calculating the extremum in Eq. (67) with respect to $\nu$ that occurs at:

$$
\nu = 3 \frac{(1 + n)^2}{(2 + 3n)}.
$$

(70)

Thus, the approximate analytical expression for the breakup criterion can be obtained by combining Eqs. (66), (67) and (70) to give:

$$
f_{\text{max}} = \text{Ca}^{\nu} \frac{1}{(3 + 1/n)^{1/3}} \times \left( \frac{(2 + 3n)^{2+3n}}{(1 + n)^{2+3n}(3 + 1/n)^{3}} \right)^{1/3}.
$$

(71)

A summary of the results obtained from the approximated analytical solution, at the breakup point, is listed in Table 2.

Upon comparing Tables 1 and 2 we find a remarkable agreement, especially for the maximum value of $f$, i.e. the breakup criterion. It is interesting to note that the values of $f_{\text{max}}$ are

Table 2

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\nu$</th>
<th>$f$</th>
<th>$L^*/\text{Ca}^{\nu}$</th>
<th>$A^*/\text{Ca}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.70</td>
<td>1.08</td>
<td>0.678</td>
<td>19.5</td>
</tr>
<tr>
<td>0.5</td>
<td>1.03</td>
<td>0.172</td>
<td>0.655</td>
<td>22.0</td>
</tr>
<tr>
<td>0.75</td>
<td>2.16</td>
<td>0.160</td>
<td>0.641</td>
<td>25.2</td>
</tr>
<tr>
<td>1</td>
<td>2.4</td>
<td>0.148</td>
<td>0.650</td>
<td>28.8</td>
</tr>
</tbody>
</table>
Fig. 8. The value of $f_{\text{max}}$ as a function of $n$. The solid line is the approximated analytical solution. The square symbols are the exact numerical calculations. The dashed line is Eq. (72).

not very sensitive to the values of $n$. For the exact numerical solution, they are in the range of 0.148–0.199 for $0 \leq n \leq 1$. The approximate analytical solution yields a maximum value of $f_{\text{max}} = 0.210$ when $n = 0$. One concludes that $f_{\text{max}}$ changes only mildly in the shear thinning range. We have plotted $f_{\text{max}}$ as a function of $n$ in Fig. 8. The solid line is the approximation while the exact numerical calculation is given by the square dots. A remarkable linear dependence appears in the range of $0.25 \leq n \leq 1$.

and it is represented by the dashed line in Fig. 8. This result should prove most useful for practical estimates of breakup criteria.

Finally, we show in Fig. 9 the dependence of the critical capillary number at the breakup point on the viscosity ratio as obtained from the approximate two-term solution. The result shows that for the same viscosity ratio, the critical capillary number increases as the power-law index $n$ decreases. Unfortunately, the experiments of Milliken and Leal [17] that involved also high deformations of drops in an extensional flow were conducted with polymeric drops that were made from aqueous solutions where surface contamination is common, resulting in tip-streaming [27]. Indeed, in a consequent report, Tretheway and Leal [13] repeated their experiments with a polymer drop that was not water based. As expected no tip-streaming was observed. However, since their new results were in the range of small to medium deformations, a direct comparison with our theory cannot be made. Thus, our calculations are still waiting for experimental corroboration.

6. Conclusions

The deformation and breakup criteria of a power-law non-Newtonian slender drop in a Newtonian liquid in a simple extensional creeping flow has been studied. The deformation of the drop is described by a single ordinary differential equation, which was numerically solved. The deformation of the drop is described by a single ordinary differential equation, which was numerically solved. Exact analytical expressions for the local radius near the center and close to the end of the drop are presented. The solution near the end suggests that a drop can exist only for cases where $n \leq 1$. As in the case of a Newtonian drop, the non-Newtonian slender drop has pointed ends. Contrary to a Newtonian drop, the predicted shape near the center of the drop is not a parabola, however, the analyticity there may be satisfied by an inner solution of the flow at the center of the drop that is not presented in this study. The numerical procedure, which was facilitated by expressing the governing equation as a power series, suggests the existence of multiple solutions. Finally, the deformation results show that Newtonian drops are more elongated than shear thinning drops. However, for the same viscosity ratio, shear thinning drops seem to be more difficult to break than Newtonian drops. An approximate analytical solution is also presented, which is in good agreement with the numerical results up to the point of maximum deformation, provides useful tools for the estimate of critical parameters at the breakup point.

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References